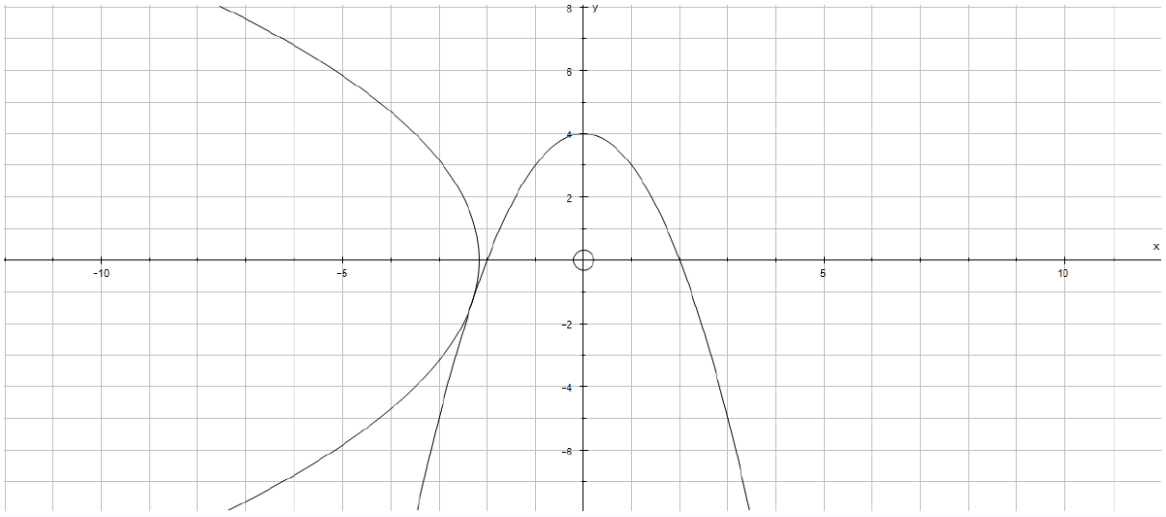
STEP III 2012 Question 3 — Mark Scheme

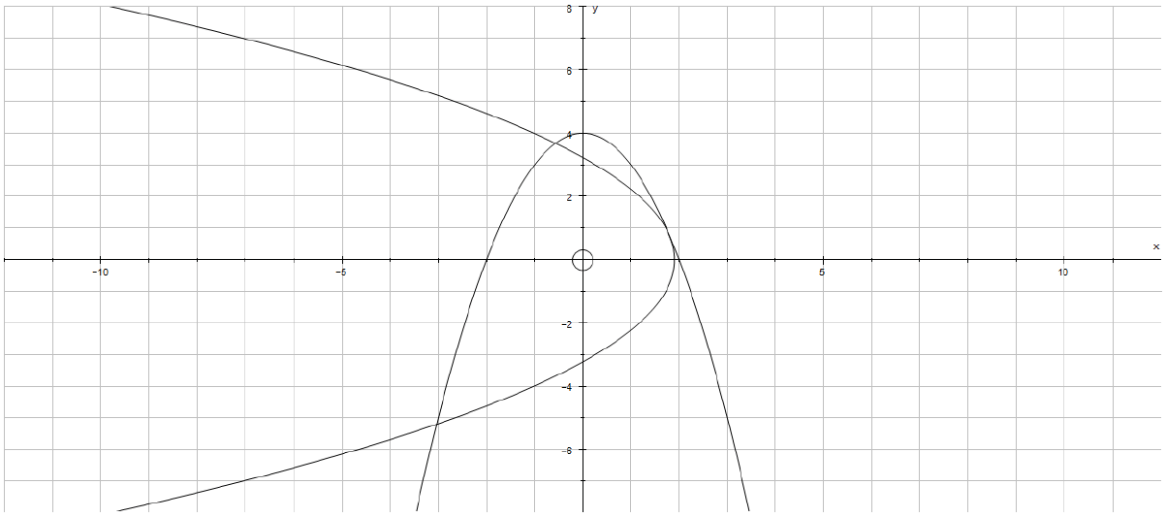
(i) **G1** for $y = 4 - x^2$ curve **B1** for all intercepts $(\pm 2, 0)$, $(0, 4)$, $\left(\frac{k}{m}, 0\right)$ at least once and $\left(0, \pm\sqrt{k}\right)$ at least once in b,c, or d

(a)



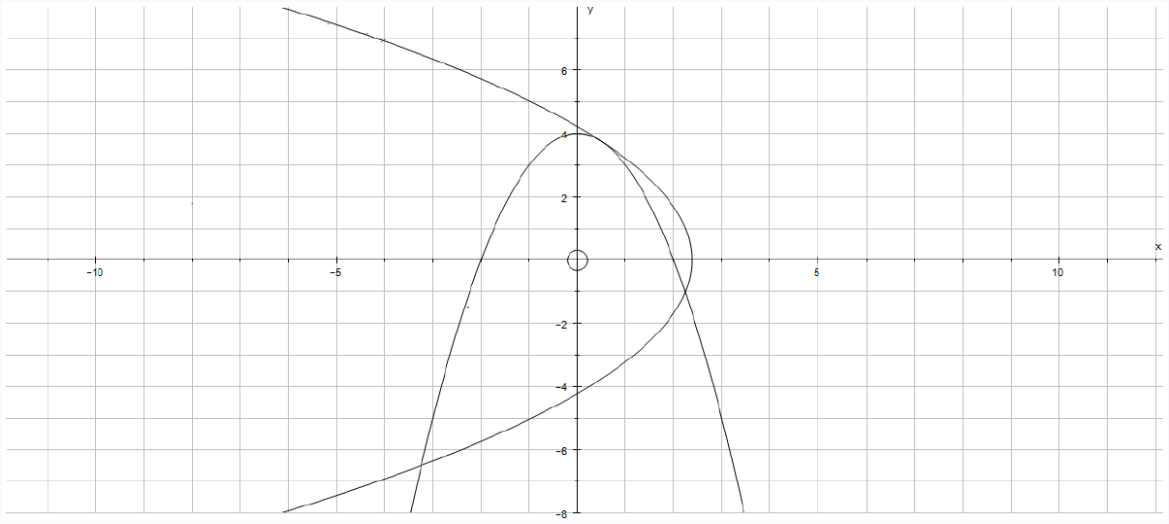
G1 correct look

(b)



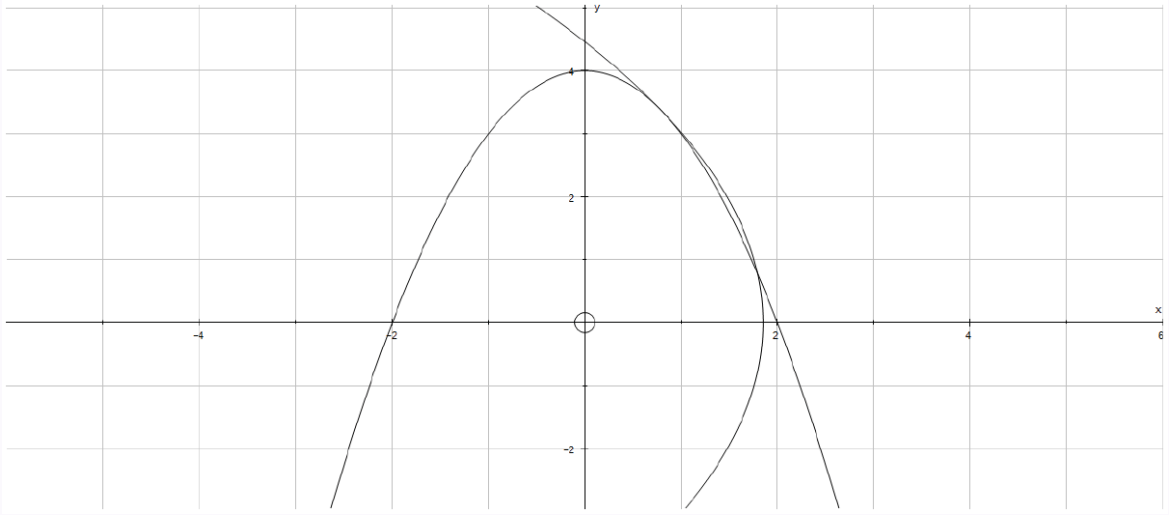
G1 correct look

(c)



G1 correct look

(d)



G1 correct look – note correct intersection and touching arrangement **(6)**

(ii) $y = 4 - x^2$ and $12x = k - y^2$

So $12x = k - (4 - x^2)^2$ at any intersection. **M1**

$$12x = k - 16 + 8x^2 - x^4$$

$$x^4 - 8x^2 + 12x + 16 - k = 0 \qquad (*) \qquad \textbf{A1}$$

If the curves touch, they have equal gradients.

$\dfrac{dy}{dx} = -2x$ and $12 = -2y\dfrac{dy}{dx}$ thus $-2x = \dfrac{-6}{y}$ **M1**

i.e. $xy = 3$ so as $x = a$, $y = 4 - a^2$

$$a(4 - a^2) = 3$$

$$a^3 - 4a + 3 = 0 \qquad \textbf{M1} \qquad (*)\ \textbf{A1}$$

(These three marks can be obtained by considering properties of the roots of the quartic, where the roots are a,a,c,d)

$$(a - 1)(a^2 + a - 3) = 0$$

So $a = 1$ or $a = \dfrac{-1 \pm \sqrt{13}}{2}$ **M1 A1**

As $x^4 - 8x^2 + 12x + 16 - k = 0$, $a^4 - 8a^2 + 12a + 16 - k = 0$, so $k = a^4 - 8a^2 + 12a + 16$

$$k = aa^3 - 8a^2 + 12a + 16 = a(4a - 3) - 8a^2 + 12a + 16 = -4a^2 + 9a + 16 \qquad \textbf{M1 (*) A1 (9)}$$

If $a = 1$, $k = 21$, $\dfrac{k}{m} = \dfrac{21}{12} < 2$ which is case (d). **B1**

$$x^4 - 8x^2 + 12x - 5 = 0,\ (x - 1)^2(x^2 + 2x - 5) = 0 \implies x = 1,\ -1 \pm \sqrt{6}$$

If $a = \dfrac{-1 + \sqrt{13}}{2}$, $k = -\left(-1 + \sqrt{13}\right)^2 + 9\left(\dfrac{-1 + \sqrt{13}}{2}\right) + 16 = \dfrac{13}{2}\sqrt{13} - \dfrac{5}{2}$, so, $\dfrac{k}{m} < 2$ which is case (d).

B2

$$x^4 - 8x^2 + 12x + \frac{37}{2} - \frac{13}{2}\sqrt{13} = 0,\ \left(x - \frac{-1 + \sqrt{13}}{2}\right)^2\left(x^2 + \left(\sqrt{13} - 1\right)x + \frac{-3\sqrt{13} + 5}{2}\right) = 0$$

$\Rightarrow x = \dfrac{-1 + \sqrt{13}}{2}$, etc

If $a = \dfrac{-1 - \sqrt{13}}{2}$, $k = -\left(-1 - \sqrt{13}\right)^2 + 9\left(\dfrac{-1 - \sqrt{13}}{2}\right) + 16 = \dfrac{-13}{2}\sqrt{13} - \dfrac{5}{2}$, so, $k < 0$ which is case (a)

B2
(5)

SCAN OR CLICK QUESTION SCAN OR CLICK EXAMINER'S COMMENTS