STEP III, 2012 Q2 MS

2. The simplification in the opening is $\left(1-x^{2^{n+1}}\right)$, obtained by repeated use of the difference of two squares. A simple algebraic rearrangement, followed by taking a limit, the logarithm of both expressions, and differentiation produces the other three results in part (i). Part (ii) can be obtained by replacing x by x^3 in $\ln(1-x) = -\sum_{r=0}^{\infty} \ln\left(1+x^{2^r}\right)$ from part (i), factorising the difference and the sums of cubes and subtracting that part (i) result before differentiating. An alternative is to replicate part (i) using instead the product

$(1+x+x^2)(1-x+x^2)(1-x^2+x^4)(1-x^4+x^8)\ldots\left(1-x^{2^n}+x^{2^{n+1}}\right)$, but then a little extra care is required with the rearrangement, and consideration of the limit.