STEP III, 2012 Q2

2 In this question, $|x| < 1$ and you may ignore issues of convergence.

(i) Simplify

$$(1-x)(1+x)(1+x^2)(1+x^4)\cdots(1+x^{2^n}),$$

where n is a positive integer, and deduce that

$$\frac{1}{1-x} = (1+x)(1+x^2)(1+x^4)\cdots(1+x^{2^n}) + \frac{x^{2^{n+1}}}{1-x}.$$

Deduce further that

$$\ln(1-x) = -\sum_{r=0}^{\infty} \ln\left(1+x^{2^r}\right),$$

and hence that

$$\frac{1}{1-x} = \frac{1}{1+x} + \frac{2x}{1+x^2} + \frac{4x^3}{1+x^4} + \cdots.$$

(ii) Show that

$$\frac{1+2x}{1+x+x^2} = \frac{1-2x}{1-x+x^2} + \frac{2x-4x^3}{1-x^2+x^4} + \frac{4x^3-8x^7}{1-x^4+x^8} + \cdots.$$