STEP III, 2012 Q12

12 **(i)** A point P lies in an equilateral triangle ABC of height 1. The perpendicular distances from P to the sides AB, BC and CA are x_1, x_2 and x_3, respectively. By considering the areas of triangles with one vertex at P, show that $x_1 + x_2 + x_3 = 1$.

Suppose now that P is placed at random in the equilateral triangle (so that the probability of it lying in any given region of the triangle is proportional to the area of that region). The perpendicular distances from P to the sides AB, BC and CA are random variables X_1, X_2 and X_3, respectively. In the case $X_1 = \min(X_1, X_2, X_3)$, give a sketch showing the region of the triangle in which P lies.

Let $X = \min(X_1, X_2, X_3)$. Show that the probability density function for X is given by

$$\mathrm{f}(x) = \begin{cases} 6(1-3x) & 0 \leqslant x \leqslant \frac{1}{3}\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

Find the expected value of X.

(ii) A point is chosen at random in a regular tetrahedron of height 1. Find the expected value of the distance from the point to the closest face.
[The volume of a tetrahedron is $\frac{1}{3} \times$ area of base $\times$ height and its centroid is a distance $\frac{1}{4} \times$ height from the base.]