STEP III, 2011 Q8 MS

8. $w = u + iv = \frac{1+i(x+iy)}{i+(x+iy)} = \frac{2x}{x^2+(1+y)^2} + i\frac{x^2-(1-y^2)}{x^2+(1+y)^2}$ using the complex conjugate, so $u = \frac{2x}{x^2+(1+y)^2}$ and $v = \frac{x^2-(1-y^2)}{x^2+(1+y)^2}$

(i) If $x = \tan\frac{\theta}{2}$, $y = 0$, then $u = \sin\theta$, and $v = -\cos\theta$, using the general result and so $u^2 + v^2 = 1$ but the point $\theta = \pi$ i.e. (0,1) is not included.

(ii) If $-1 < x < 1$, and $y = 0$, then it is the same locus as (i) except $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, and so it is the semi-circle that is the part of $u^2 + v^2 = 1$ below the u axis.

(iii) $x = 0$, then $u = 0$ and $v = \frac{y-1}{y+1}$, and as $-1 < y < 1$ which is that part of the v axis below the u axis, i.e. $-\infty < v < 0$.

(iv) Let $x = 2\tan\frac{\theta}{2}$ and $y = 1$, so as $-\infty < x < \infty$, $-\pi < \theta < \pi$, then $u = \frac{1}{2}\sin\theta$ and $v = \frac{1}{2}(1 - \cos\theta)$, so the locus is the circle $u^2 + \left(v - \frac{1}{2}\right)^2 = \left(\frac{1}{2}\right)^2$ excluding the point $\theta = \pi$, which is (0,1).