STEP III, 2011 Q7 MS

7. (i) The induction requires $T_{k+2} = A_{k+2} + B_{k+2}\sqrt{a(a+1)}$ and $A_{k+2}{}^2 - a(a+1)B_{k+2}{}^2 = 1$.

$T_{k+2} = \left(A_k + B_k\sqrt{a(a+1)}\right)\left(\sqrt{a+1} + \sqrt{a}\right)^2 = \left(A_k + B_k\sqrt{a(a+1)}\right)T_2$

$T_2 = \left(2a + 1 + 2\sqrt{a(a+1)}\right)$ and so $A_2 = 2a + 1$ and $B_2 = 2$, and $A_2{}^2 - a(a+1)B_2{}^2 = (2a+1)^2 - a(a+1)2^2 = 1$ the result is true for $n = 2$.

Evaluating T_{k+2} using $\left(A_k + B_k\sqrt{a(a+1)}\right)T_2$ then $A_{k+2} = (2a+1)A_k + 2a(a+1)B_k$ and $B_{k+2} = 2A_k + (2a+1)B_k$, and so substituting and simplifying, $A_{k+2}{}^2 - a(a+1)B_{k+2}{}^2 = A_k{}^2 - a(a+1)B_k{}^2 = 1$ by the induction.

(ii) $T_n = \left(\sqrt{a+1} + \sqrt{a}\right)T_m = \left(\sqrt{a+1} + \sqrt{a}\right)\left(A_m + B_m\sqrt{a(a+1)}\right)$

$= (A_m + aB_m)\sqrt{a+1} + (A_m + (a+1)B_m)\sqrt{a}$ which is of required form because

$C_n = A_m + aB_m$ and $D_n = A_m + (a+1)B_m$ are integers and

$(a+1)C_n{}^2 - aD_n{}^2 = (a+1)(A_m + aB_m)^2 - a(A_m + (a+1)B_m)^2$

$= A_m{}^2 - a(a+1)B_m{}^2 = 1$ as required.

Trivially the case $n = 1$ is true.

(iii) In the case that n is even,

$T_n = A_n + B_n\sqrt{a(a+1)} = \sqrt{A_n{}^2} + \sqrt{a(a+1)B_n{}^2} = \sqrt{a(a+1)B_n{}^2 + 1} + \sqrt{a(a+1)B_n{}^2}$ as required,

and in the case that n is odd, $T_n = C_n\sqrt{a+1} + D_n\sqrt{a} = \sqrt{(a+1)C_n{}^2} + \sqrt{aD_n{}^2} = \sqrt{aD_n{}^2 + 1} + \sqrt{aD_n{}^2}$ as required.