

STEP III, 2011 Q7 MS

7. (i) The induction requires $T_{k+2} = A_{k+2} + B_{k+2}\sqrt{a(a+1)}$ and $A_{k+2}^2 - a(a+1)B_{k+2}^2 = 1$.

$$T_{k+2} = (A_k + B_k\sqrt{a(a+1)})(\sqrt{a+1} + \sqrt{a})^2 = (A_k + B_k\sqrt{a(a+1)})T_2$$

$T_2 = (2a + 1 + 2\sqrt{a(a+1)})$ and so $A_2 = 2a + 1$ and $B_2 = 2$, and

$A_2^2 - a(a+1)B_2^2 = (2a+1)^2 - a(a+1)2^2 = 1$ the result is true for $n = 2$.

Evaluating T_{k+2} using $(A_k + B_k\sqrt{a(a+1)})T_2$ then $A_{k+2} = (2a+1)A_k + 2a(a+1)B_k$

and $B_{k+2} = 2A_k + (2a+1)B_k$, and so substituting and simplifying,

$A_{k+2}^2 - a(a+1)B_{k+2}^2 = A_k^2 - a(a+1)B_k^2 = 1$ by the induction.

$$(ii) \quad T_n = (\sqrt{a+1} + \sqrt{a})T_m = (\sqrt{a+1} + \sqrt{a})(A_m + B_m\sqrt{a(a+1)})$$

$= (A_m + aB_m)\sqrt{a+1} + (A_m + (a+1)B_m)\sqrt{a}$ which is of required form because

$C_n = A_m + aB_m$ and $D_n = A_m + (a+1)B_m$ are integers and

$$(a+1)C_n^2 - aD_n^2 = (a+1)(A_m + aB_m)^2 - a(A_m + (a+1)B_m)^2$$

$$= A_m^2 - a(a+1)B_m^2 = 1 \text{ as required.}$$

Trivially the case $n = 1$ is true.

(iii) In the case that n is even,

$$T_n = A_n + B_n\sqrt{a(a+1)} = \sqrt{A_n^2} + \sqrt{a(a+1)B_n^2} = \sqrt{a(a+1)B_n^2 + 1} + \sqrt{a(a+1)B_n^2}$$

as required,

$$\text{and in the case that } n \text{ is odd, } T_n = C_n\sqrt{a+1} + D_n\sqrt{a} = \sqrt{(a+1)C_n^2} + \sqrt{aD_n^2} =$$

$$\sqrt{aD_n^2 + 1} + \sqrt{aD_n^2} \text{ as required.}$$



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