STEP III, 2011 Q6 MS

6. Using the substitution $t = \tanh\left(\frac{u}{2}\right)$, then it can be shown that $T = U$, by making use of $2\sinh\left(\frac{u}{2}\right)\cosh\left(\frac{u}{2}\right) = \sinh u$ to obtain the integrand, and $\tanh^{-1} t = \frac{1}{2}\ln\left(\frac{1+t}{1-t}\right)$ to obtain the limits.

If instead, integration by parts is used differentiating $\tanh^{-1} t$ and integrating $\frac{1}{t}$, and employing $\tanh^{-1} t = \frac{1}{2}\ln\left(\frac{1+t}{1-t}\right)$ to demonstrate that $[\tanh^{-1} t \ln t]_{\frac{1}{3}}^{\frac{1}{2}} = 0$, $T = V$.

The substitution $t = e^{-2x}$ can be used to demonstrate that $T = X$.

(Alternatively, starting from U, the substitution $u = 2\tanh^{-1} t$ obtains $U = T$, the substitution $u = -\ln v$ obtains $U = V$, and the substitution $u = 2x$ followed by integration by parts yields $U = X$; starting from V, by parts it can be shown that $V = T$, using the substitution $v = e^{-u}$ that $V = U$, and the substitution $v = \tanh x$ that $V = X$; or starting from X, the substitution $x = -\frac{1}{2}\ln t$ gives $X = T$, integration by parts gives $X = U$, and the substitution $x = \tanh^{-1} v$ gives $X = V$.)