STEP III, 2011 Q5 MS

5.

$r^2 d\theta = (x^2+y^2)\frac{d}{dt}\left(\tan^{-1}\left(\frac{y}{x}\right)\right)dt = (x^2+y^2)\frac{1}{1+\left(\frac{y}{x}\right)^2}\frac{\left(x\frac{dy}{dt}-y\frac{dx}{dt}\right)}{x^2}dt \; = \left(x\frac{dy}{dt}-y\frac{dx}{dt}\right)dt$

and hence integrating gives the result.

A is $(x-a\cos t, y-a\sin t)$ and B is $(x+b\cos t, y+b\sin t)$

$[A] = \frac{1}{2}\int_0^{2\pi}(x-a\cos t)\left(\frac{dy}{dt}-a\cos t\right)-(y-a\sin t)\left(\frac{dx}{dt}+a\sin t\right)dt$ using (*) which leads directly to $[A]=[P]-af+\pi a^2$.
Replacing $-a$ by b gives $[B]=[P]+bf+\pi b^2$
As $[A]=[B]$, these expressions can be equated to give $f=\pi(a-b)$.
The area between curves C and D is $[A]-[P]=-af+\pi a^2$ which by substitution gives πab as required.