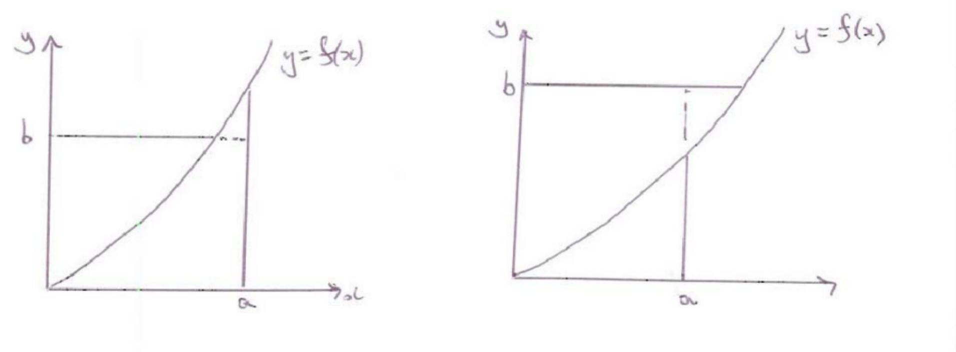
STEP III, 2011 Q4 MS

4. (i) $\int_0^a f(x)dx$ is the area between the curve $y = f(x)$, the x axis, and the line $x = a$
$\int_0^b f^{-1}(y)dy$ is the area between the curve $y = f(x)$, the y axis, and the line $y = b$.

The sum of these areas is greater than or equal to the area of the rectangle, with equality holding if $b = f(a)$.



(ii) With $(x) = x^{p-1}$, the sum of the two integrals is $\frac{1}{p}a^p + \frac{p-1}{p}b^{\frac{p}{p-1}}$

But as $\frac{1}{p} + \frac{1}{q} = 1$, $\frac{1}{q} = \frac{p-1}{p}$, and so the required result follows by applying the result of part (i).
If $b = a^{p-1}$, simple algebra shows $a = b^{q-1}$, so $\frac{1}{p}a^p + \frac{1}{q}b^q = \frac{1}{p}ab + \frac{1}{q}ba = ab$ and equality is verified.

(iii) $f(x) = \sin x$ satisfies the conditions of part (i)
So $\int_o^a f(x)\,dx = 1 - \cos a$, and, by parts, $\int_0^b f^{-1}(y)dy = b\sin^{-1} b + \sqrt{1-b^2} - 1$
which together give the required result.

Choosing $a = 0$, and $b = t^{-1}$, part (i) gives $0 \le t^{-1}\sin^{-1}(t^{-1}) + \sqrt{1-t^{-2}} - 1$ which can be re-arranged to give the required result.