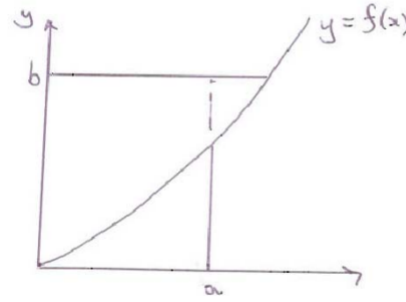
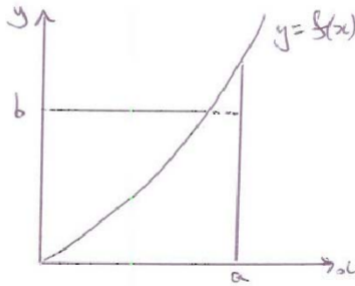


STEP III, 2011 Q4 MS

4. (i) $\int_0^a f(x) dx$ is the area between the curve $y = f(x)$, the x axis, and the line $x = a$
 $\int_0^b f^{-1}(y) dy$ is the area between the curve $y = f(x)$, the y axis, and the line $y = b$.

The sum of these areas is greater than or equal to the area of the rectangle, with equality holding if $b = f(a)$.



- (ii) With $f(x) = x^{p-1}$, the sum of the two integrals is $\frac{1}{p} a^p + \frac{p-1}{p} b^{\frac{p}{p-1}}$

But as $\frac{1}{p} + \frac{1}{q} = 1$, $\frac{1}{q} = \frac{p-1}{p}$, and so the required result follows by applying the result of part (i).

If $b = a^{p-1}$, simple algebra shows $a = b^{q-1}$, so $\frac{1}{p} a^p + \frac{1}{q} b^q = \frac{1}{p} ab + \frac{1}{q} ba = ab$ and equality is verified.

- (iii) $f(x) = \sin x$ satisfies the conditions of part (i)

So $\int_0^a f(x) dx = 1 - \cos a$, and, by parts, $\int_0^b f^{-1}(y) dy = b \sin^{-1} b + \sqrt{1 - b^2} - 1$ which together give the required result.

Choosing $a = 0$, and $b = t^{-1}$, part (i) gives $0 \leq t^{-1} \sin^{-1}(t^{-1}) + \sqrt{1 - t^{-2}} - 1$ which can be re-arranged to give the required result.



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