STEP III, 2011 Q4

4 The following result applies to any function f which is continuous, has positive gradient and satisfies $\mathrm{f}(0) = 0$:

$$ab \leqslant \int_0^a \mathrm{f}(x)\,\mathrm{d}x + \int_0^b \mathrm{f}^{-1}(y)\,\mathrm{d}y\,, \qquad (*)$$

where f^{-1} denotes the inverse function of f, and $a \geqslant 0$ and $b \geqslant 0$.

(i) By considering the graph of $y = \mathrm{f}(x)$, explain briefly why the inequality $(*)$ holds.

In the case $a > 0$ and $b > 0$, state a condition on a and b under which equality holds.

(ii) By taking $\mathrm{f}(x) = x^{p-1}$ in $(*)$, where $p > 1$, show that if $\dfrac{1}{p} + \dfrac{1}{q} = 1$ then

$$ab \leqslant \frac{a^p}{p} + \frac{b^q}{q}\,.$$

Verify that equality holds under the condition you stated above.

(iii) Show that, for $0 \leqslant a \leqslant \frac{1}{2}\pi$ and $0 \leqslant b \leqslant 1$,

$$ab \leqslant b \arcsin b + \sqrt{1-b^2} - \cos a\,.$$

Deduce that, for $t \geqslant 1$,

$$\arcsin(t^{-1}) \geqslant t - \sqrt{t^2 - 1}\,.$$