

STEP III, 2011 Q4

- 4 The following result applies to any function f which is continuous, has positive gradient and satisfies $f(0) = 0$:

$$ab \leq \int_0^a f(x) dx + \int_0^b f^{-1}(y) dy, \quad (*)$$

where f^{-1} denotes the inverse function of f , and $a \geq 0$ and $b \geq 0$.

- (i) By considering the graph of $y = f(x)$, explain briefly why the inequality $(*)$ holds.
In the case $a > 0$ and $b > 0$, state a condition on a and b under which equality holds.
- (ii) By taking $f(x) = x^{p-1}$ in $(*)$, where $p > 1$, show that if $\frac{1}{p} + \frac{1}{q} = 1$ then

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

Verify that equality holds under the condition you stated above.

- (iii) Show that, for $0 \leq a \leq \frac{1}{2}\pi$ and $0 \leq b \leq 1$,

$$ab \leq b \arcsin b + \sqrt{1 - b^2} - \cos a.$$

Deduce that, for $t \geq 1$,

$$\arcsin(t^{-1}) \geq t - \sqrt{t^2 - 1}.$$



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