STEP III, 2011 Q3 MS

3. Considering the quadratic equation $pt^2 - qt + p^2 = 0$, the condition $q^2 \neq 4p^3$ shows, by considering the discriminant, that the roots are unequal. Supposing that $x^3 - 3px + q$ can be written as $a(x-\alpha)^3 + b(x-\beta)^3$, and equating coefficients generates the four equations

$a + b = 1$

$-3\alpha a - 3\beta b = 0$

$3\alpha^2 a + 3\beta^2 b = -3p$

$-\alpha^3 a - \beta^3 b = q$

The first pair may be solved simultaneously to give $a = \frac{-\beta}{\alpha-\beta}$ and $b = \frac{\alpha}{\alpha-\beta}$.
Substitution yields $p = \alpha\beta$ and $q = \alpha\beta(\alpha+\beta)$, or alternatively,
$\alpha\beta = p$ and $\alpha + \beta = \frac{q}{p}$ and so α and β satisfy $t^2 - \frac{q}{p}t + p = 0$ i.e. $pt^2 - qt + p^2 = 0$.
For $p = 8,\ q = 48$, $q^2 - 4p^3 = 2^8 \neq 0$.
Hence α and β are the roots of $8t^2 - 48t + 64 = 0$, i.e. $t^2 - 6t + 8 = 0$ and wlog $\alpha = 2,\ \beta = 4,\ a = 2,\ b = -1$.
So $x^3 - 24x + 48 = 0$ can be re-arranged as $\left(\frac{x-4}{x-2}\right)^3 = 2$
As $\omega^3 = 1$, $\frac{x-4}{x-2} = \sqrt[3]{2},\ \omega\sqrt[3]{2},\ \omega^2\sqrt[3]{2}$ and so $x = \frac{2\left(2-\sqrt[3]{2}\right)}{1-\sqrt[3]{2}}, \frac{2\left(2-\omega\sqrt[3]{2}\right)}{1-\omega\sqrt[3]{2}}, \frac{2\left(2-\omega^2\sqrt[3]{2}\right)}{1-\omega^2\sqrt[3]{2}}$

If $q = 2r^3$ and $p = r^2$, $q^2 = 4p^3$ so the first part cannot be used.
However, $x^3 - 3r^2x + 2r^3 = 0$ can be readily factorised as $(x-r)^2(x+2r) = 0$ and so $x = r$ (repeated) or $-2r$