

STEP III, 2011 Q3 MS

3. Considering the quadratic equation $pt^2 - qt + p^2 = 0$, the condition $q^2 \neq 4p^3$ shows, by considering the discriminant, that the roots are unequal. Supposing that $x^3 - 3px + q$ can be written as $a(x - \alpha)^3 + b(x - \beta)^3$, and equating coefficients generates the four equations

$$\begin{aligned} a + b &= 1 \\ -3\alpha a - 3\beta b &= 0 \\ 3\alpha^2 a + 3\beta^2 b &= -3p \\ -\alpha^3 a - \beta^3 b &= q \end{aligned}$$

The first pair may be solved simultaneously to give $a = \frac{-\beta}{\alpha - \beta}$ and $b = \frac{\alpha}{\alpha - \beta}$.

Substitution yields $p = \alpha\beta$ and $q = \alpha\beta(\alpha + \beta)$, or alternatively,

$$\alpha\beta = p \text{ and } \alpha + \beta = \frac{q}{p} \text{ and so } \alpha \text{ and } \beta \text{ satisfy } t^2 - \frac{q}{p}t + p = 0 \text{ i.e. } pt^2 - qt + p^2 = 0.$$

For $p = 8$, $q = 48$, $q^2 - 4p^3 = 2^8 \neq 0$.

Hence α and β are the roots of $8t^2 - 48t + 64 = 0$, i.e. $t^2 - 6t + 8 = 0$ and wlog $\alpha = 2$, $\beta = 4$, $a = 2$, $b = -1$.

So $x^3 - 24x + 48 = 0$ can be re-arranged as $\left(\frac{x-4}{x-2}\right)^3 = 2$

$$\text{As } \omega^3 = 1, \quad \frac{x-4}{x-2} = \sqrt[3]{2}, \omega\sqrt[3]{2}, \omega^2\sqrt[3]{2} \text{ and so } x = \frac{2(2-\sqrt[3]{2})}{1-\sqrt[3]{2}}, \frac{2(2-\omega\sqrt[3]{2})}{1-\omega\sqrt[3]{2}}, \frac{2(2-\omega^2\sqrt[3]{2})}{1-\omega^2\sqrt[3]{2}}$$

If $q = 2r^3$ and $p = r^2$, $q^2 = 4p^3$ so the first part cannot be used.

However, $x^3 - 3r^2x + 2r^3 = 0$ can be readily factorised as $(x - r)^2(x + 2r) = 0$ and so $x = r$ (repeated) or $-2r$



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