STEP III, 2011 Q2 MS

2. As $f\left(\frac{p}{q}\right) = 0$, $q^{n-1} f\left(\frac{p}{q}\right) = 0$, which, when evaluated, gives every term but one to be an integer, and so, that term, $\frac{p^n}{q}$, must be an integer, and as p and q are integers with no common factor greater than 1, this can only happen if $q = 1$, giving the required deduction.

(i) To show that the nth root of 2 is irrational, consider $f(x) = x^n - 2$, and evaluate $f(1)$ and $f(2)$, then apply the stem of the question.

(ii) Considering the turning points of $f(x) = x^3 - x + 1$, there can only be one real root. Evaluating $f(-2)$ and $f(-1)$ and applying the stem gives the result.

(iii) Considering the graphs of $y = x^n$ and $y = 5x - 7$, for $n \geq 3$, that these cannot intersect for $x \geq 0$ can be observed by noting their signs for $0 \leq x < 1 \cdot 4$, and their gradients for $x \geq 1 \cdot 4$. For $x < 0$, and n even, it is sufficient to consider signs, whereas for n odd, it is enough to evaluate $f(x) = x^n - 5x + 7$ for $x = -2$, and $x = -1$ $or -3$, depending on the case, and then applying the stem. The case $n = 2$, can be demonstrated by completing the square and showing that there are no real roots.

Part (i) could be demonstrated by a minor variant to the usual proof for the irrationality of the square root of 2. Parts (ii) and (iii) could be shown by applying the stem and then considering the left hand side of each equation for the cases n even and n odd.

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