STEP III, 2011 Q1 MS

1. (i) The differential equation can be solved either by separating variables or using an integrating factor. In either case, $\int \left(\frac{x+2}{x+1}\right) dx$, or the negative of it is required, and this can be found either by re-writing $\left(\frac{x+2}{x+1}\right)$ as $1 + \frac{1}{x+1}$ or using the substitution, $v = x + 1$. Thus the solution is $u = k(x+1)e^x$.

(ii) The substitution $y = ze^{-x}$ yields $\frac{dy}{dx} = z'e^{-x} - ze^{-x}$, and $\frac{d^2y}{dx^2} = z''e^{-x} - 2z'e^{-x} + ze^{-x}$.

Substituting these expressions in the differential equation and simplifying gives

$\left((x+1)z'' - (x+2)z'\right)e^{-x} = 0$ which is effectively the first order differential equation from part (i) with $u = z'$.

So $z' = k(x+1)e^x$, which is an exact differential (or integration by parts could be used), $z = kxe^x + c$ and so $y = Ax + Be^{-x}$ as required.

(iii) Part (ii)'s substitution gives $z'' - \frac{(x+2)}{(x+1)}z' = (x+1)e^x$ which using the integrating factor from part (i) gives $\frac{e^{-x}}{x+1}z' = \int 1dx = x + c$, and thus

$y = (x^2+1) + Ax + Be^{-x}$. Alternatively, the solution to part (ii) is the complementary function and a quadratic particular integral should be conjectured, which in view of the cf need only be $y = Cx^2 + D$, yielding the same result.