STEP III, 2011 Q12 MS

12. As $G_Y(t) = G(H(t))$, $G'_Y(t) = G'(H(t)) \times H'(t)$, and as $H(1) = 1$, $H'(1) = E(X_i)$, $G'(1) = E(N)$, and $G'_Y(1) = E(Y)$, the first result follows.

Similarly, $G''_Y(t) = G''(H(t)) \times (H'(t))^2 + G'(H(t)) \times H''(t)$, and

$Var(Y) = G''_Y(1) + G'_Y(1) - (G'_Y(1))^2$

$= G''(H(1)) \times (H'(1))^2 + G'(H(1)) \times H''(1) + E(Y) - (E(Y))^2$

$= \left(Var(N) + (E(N))^2 - E(N)\right) \times (E(X_i))^2 + E(N) \times \left(Var(X_i) + (E(X_i))^2 - E(X_i)\right) + E(N)\,E(X_i) - (E(N)\,E(X_i))^2$

$= Var(N) \times (E(X_i))^2 + E(N) \times Var(X_i)$ as required.

A fair coin tossed until a head appears is distributed $Geo\left(\frac{1}{2}\right)$ so $G(t) = \frac{t}{2-t}$. The PGF for the number of heads when a fair coin is tossed once is $\frac{1}{2} + \frac{1}{2}t$. Thus $G_Y(t) = \frac{1+t}{3-t}$.

Using the results $E(Y) = 2 \times \frac{1}{2} = 1$, and $Var(Y) = \frac{1-\frac{1}{2}}{\left(\frac{1}{2}\right)^2} \times \left(\frac{1}{2}\right)^2 + 2 \times \frac{1}{4} = 1$.

$P(Y = r)$, being the coefficient of t^r in $G_Y(t)$, is $\frac{4}{3^{r+1}}$ for $r \geq 1$, and $\frac{1}{3}$ for $r = 0$.