STEP III, 2011 Q11

11 A thin uniform circular disc of radius a and mass m is held in equilibrium in a horizontal plane a distance b below a horizontal ceiling, where $b > 2a$. It is held in this way by n light inextensible vertical strings, each of length b; one end of each string is attached to the edge of the disc and the other end is attached to a point on the ceiling. The strings are equally spaced around the edge of the disc. One of the strings is attached to the point P on the disc which has coordinates $(a, 0, -b)$ with respect to cartesian axes with origin on the ceiling directly above the centre of the disc.

The disc is then rotated through an angle θ (where $\theta < \pi$) about its vertical axis of symmetry and held at rest by a couple acting in the plane of the disc. Show that the string attached to P now makes an angle ϕ with the vertical, where

$$b \sin \phi = 2a \sin \tfrac{1}{2}\theta \,.$$

Show further that the magnitude of the couple is

$$\frac{mga^2 \sin \theta}{\sqrt{b^2 - 4a^2 \sin^2 \tfrac{1}{2}\theta}} \,.$$

The disc is now released from rest. Show that its angular speed, ω, when the strings are vertical is given by

$$\frac{a^2\omega^2}{4g} = b - \sqrt{b^2 - 4a^2 \sin^2 \tfrac{1}{2}\theta} \,.$$