STEP III, 2010 Q8 MS

8. Substituting for $P(x)$, the desired integral is seen to be the reverse of the quotient rule, i.e.

$$\frac{R(x)}{Q(x)}(+k)$$

To choose a suitable function $R(x)$ in part (i), substitution of $R(x) = a + bx + cx^2$ and $Q(x) = 1 + 2x + 3x^2$ in the given expression yields a quadratic equation, and equating the coefficients of the powers of x gives $5 = -3b + 2c$, $-2 = -3a + c$, $-3 = -2a + b$.
These three equations are linearly dependent and so their solution is not unique.
Choosing, for example $a = 0$, $= -3$, $c = -2$ and then $a = 1$, $b = -1$, $c = 1$ gives solutions which are related by $\frac{1-x+x^2}{1+2x+3x^2} = \frac{1+2x+3x^2-3x-2x^2}{1+2x+3x^2} = 1 + \frac{-3x-2x^2}{1+2x+3x^2}$ i.e. the same bar the

arbitrary constant.

(ii) Rearranging the equation to be solved as $\frac{dy}{dx} + \frac{(\sin x - 2\cos x)}{(1+\cos x+2\sin x)}y = \frac{(5-3\cos x+4\sin x)}{(1+\cos x+2\sin x)}$, the integrating factor is $e^{\int\frac{(\sin x-2\cos x)}{(1+\cos x+2\sin x)}dx} = e^{-\ln(1+\cos x+2\sin x)} = \frac{1}{1+\cos x+2\sin x}$
As a result, the RHS we require to integrate is $\frac{(5-3\cos x+4\sin x)}{(1+\cos x+2\sin x)^2}$
Repeating similar working to part (i), except with $Q(x) = 1 + \cos x + 2\sin x$ and $R(x) = a + b\sin x + c\cos x$, gives three linearly dependent equations,
$5 = b - 2c$, $-3 = b - 2a$, $4 = a - c$
Choosing e.g. $= 4$, $b = 5$, $c = 0$, the solution is $y = 4 + 5\sin x + k(1 + \cos x + 2\sin x)$