STEP III, 2010 Q7 MS

7. The initial result can be obtained by differentiating y directly twice obtaining

$\frac{dy}{dx} = -\sin(m\sin^{-1}x)\frac{m}{\sqrt{1-x^2}}$

$\frac{d^2y}{dx^2} = -\cos(m\sin^{-1}x)\frac{m^2}{1-x^2} - \sin(m\sin^{-1}x)\frac{mx}{(1-x^2)^{\frac{3}{2}}}$ and substituting into the LHS.

(Slightly more elegant is to rearrange as $\cos^{-1}y = m\sin^{-1}x$, differentiate and then square to obtain $(1-x^2)\left(\frac{dy}{dx}\right)^2 = m^2(1-y^2)$ and then differentiate a second time.)

The two similar results are $(1-x^2)\frac{d^3y}{dx^3} - 3x\frac{d^2y}{dx^2} + (m^2-1)\frac{dy}{dx} = 0$ and

$(1-x^2)\frac{d^4y}{dx^4} - 5x\frac{d^3y}{dx^3} + (m^2-4)\frac{d^2y}{dx^2} = 0$, which lead to the conjecture

$(1-x^2)\frac{d^{n+2}y}{dx^{n+2}} - (2n+1)x\frac{d^{n+1}y}{dx^{n+1}} + (m^2-n^2)\frac{d^ny}{dx^n} = 0$ which is proved simply by induction.

Using $= 0$, we find that $y = 1$, $\frac{dy}{dx} = 0$, $\frac{d^2y}{dx^2} = -m^2$, $\frac{d^3y}{dx^3} = 0$, $\frac{d^4y}{dx^4} = m^2(m^2-4)$

and so the Maclaurin series commences $y = 1 - \frac{m^2}{2!}x^2 + \frac{m^2(m^2-2^2)}{4!}x^4 + \cdots$

Now replacing x by $\sin\theta$,

$\cos m\theta = 1 - \frac{m^2}{2!}x^2 + \frac{m^2(m^2-2^2)}{4!}x^4 + \cdots = 1 - \frac{m^2}{2!}\sin^2\theta + \frac{m^2(m^2-2^2)}{4!}\sin^4\theta + \cdots$

All the odd differentials are zero, and the even ones are $(-1)^{k+1}m^2(m^2-2^2)\ldots(m^2-(2k)^2)$, so if m is even all the terms are zero from a certain point (when $m = 2k$) and thus the series terminates and is a polynomial in $\sin\theta$, of degree m.