STEP III, 2010 Q7

7 Given that $y = \cos(m \arcsin x)$, for $|x| < 1$, prove that

$$(1 - x^2)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - x\frac{\mathrm{d}y}{\mathrm{d}x} + m^2y = 0\,.$$

Obtain a similar equation relating $\dfrac{\mathrm{d}^3y}{\mathrm{d}x^3}$, $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$ and $\dfrac{\mathrm{d}y}{\mathrm{d}x}$, and a similar equation relating $\dfrac{\mathrm{d}^4y}{\mathrm{d}x^4}$, $\dfrac{\mathrm{d}^3y}{\mathrm{d}x^3}$ and $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$.

Conjecture and prove a relation between $\dfrac{\mathrm{d}^{n+2}y}{\mathrm{d}x^{n+2}}$, $\dfrac{\mathrm{d}^{n+1}y}{\mathrm{d}x^{n+1}}$ and $\dfrac{\mathrm{d}^ny}{\mathrm{d}x^n}$.

Obtain the first three non-zero terms of the Maclaurin series for y. Show that, if m is an even integer, $\cos m\theta$ may be written as a polynomial in $\sin\theta$ beginning

$$1 - \frac{m^2\sin^2\theta}{2!} + \frac{m^2(m^2 - 2^2)\sin^4\theta}{4!} - \cdots\,. \qquad (|\theta| < \tfrac{1}{2}\pi)$$

State the degree of the polynomial.