# STEP III, 2010 Q5 MS

5. The line CP can be shown to have equation $(1-n)y = x - an$ and so R is $\left(0, \frac{an}{n-1}\right)$

So, similarly, S must be $\left(\frac{am}{m-1}, 0\right)$.

Thus RS has equation $n(m-1)x + m(n-1)y = amn$ and PQ has equation $mx + ny = amn$.
As the coordinates of T satisfy both equations, they satisfy their difference which is
$(mn - n - m)(x + y) = 0$. As RS and PQ intersect, $\frac{n}{m} \neq \frac{m(n-1)}{n(m-1)}$ which yields
$(m-n)(mn - m - n) \neq 0$ and hence $(mn - m - n) \neq 0$ implying that T's coordinates satisfy $x + y = 0$ giving the desired result. (Alternatively, $mn - m - n = 0 \Leftrightarrow n = \frac{m}{m-1} < 0$ , which is a contradiction.)

The construction can be achieved more than one way, but one is to label the given square ABCD anti-clockwise, choose points on AB and AD different distances from A, label them P and Q, construct CP and CQ, and find their intersections with AD and AB, R and S, respectively, and find the intersection of PQ and RS, label it T, then TA is perpendicular to AC. Rotating the labelling through a right angle and repeating three more times achieves the desired square.