STEP III, 2010 Q4 MS

4. (i) As α satisfies both equations, $\alpha^2 + a\alpha + b = 0$ and $\alpha^2 + c\alpha + d = 0$, so subtracting these the desired result is simply found.
If $(b-d)^2 - a(b-d)(a-c) + b(a-c)^2 = 0$, then we may divide by $(a-c)^2$, and find that $-\frac{(b-d)}{(a-c)}$ satisfies $x^2 + ax + b = 0$. But also,
$\left(\frac{(b-d)}{(a-c)}\right)^2 + c\left(-\frac{(b-d)}{(a-c)}\right) + d = \left(\frac{(b-d)}{(a-c)}\right)^2 + a\left(-\frac{(b-d)}{(a-c)}\right) + b + (c-a)\left(-\frac{(b-d)}{(a-c)}\right) + (d-b)$ and so $-\frac{(b-d)}{(a-c)}$ satisfies $x^2 + cx + d = 0$.
On the other hand if there is a common root, then it is found at the start of the question and as it satisfies $\alpha^2 + a\alpha + b = 0$, the required result is found.
If $(b-d)^2 - a(b-d)(a-c) + b(a-c)^2 = 0$ and $a = c$, then $b = d$ and so the two equations are one and trivially have a common root. Alternatively, if there is a common root and $a = c$, then the initial subtraction yields $b = d$, and so the result is trivially true.

(ii) If $(b-r)^2 - a(b-r)(a+b-q) + b(a+b-q)^2 = 0$, then $x^2 + ax + b = 0$ and $x^2 + (q-b)x + r = 0$ have a common root from (i), and so then do $x^2 + ax + b = 0$ and $x(x^2 + ax + b) + x^2 + (q-b)x + r = 0$ which is the required result.
On the other hand, if the two equations have a common root α, then $\alpha^2 + a\alpha + b = 0$ and $\alpha^3 + (a+1)\alpha^2 + q\alpha + r = 0$, and thus so does $\alpha^3 + (a+1)\alpha^2 + q\alpha + r - \alpha(\alpha^2 + a\alpha + b) = 0$ which is a quadratic equation and we can use the result from (i) again.
Using $= \frac{5}{2}, q = \frac{5}{2}, r = \frac{1}{2}$, in the given condition, we obtain a cubic equation in b, $b^3 - \frac{3}{2}b^2 + \frac{1}{4}b + \frac{1}{4} = 0$, which has a solution $b = 1$, meaning the other two can be simply obtained as $b = \frac{1 \pm \sqrt{5}}{4}$.