STEP III, 2010 Q3

3 For any given positive integer n, a number a (which may be complex) is said to be a *primitive nth root of unity* if $a^n = 1$ and there is no integer m such that $0 < m < n$ and $a^m = 1$. Write down the two primitive 4th roots of unity.

Let $\mathrm{C}_n(x)$ be the polynomial such that the roots of the equation $\mathrm{C}_n(x) = 0$ are the primitive nth roots of unity, the coefficient of the highest power of x is one and the equation has no repeated roots. Show that $\mathrm{C}_4(x) = x^2 + 1$.

(i) Find $\mathrm{C}_1(x)$, $\mathrm{C}_2(x)$, $\mathrm{C}_3(x)$, $\mathrm{C}_5(x)$ and $\mathrm{C}_6(x)$, giving your answers as unfactorised polynomials.

(ii) Find the value of n for which $\mathrm{C}_n(x) = x^4 + 1$.

(iii) Given that p is prime, find an expression for $\mathrm{C}_p(x)$, giving your answer as an unfactorised polynomial.

(iv) Prove that there are no positive integers q, r and s such that $\mathrm{C}_q(x) \equiv \mathrm{C}_r(x)\mathrm{C}_s(x)$.