STEP III, 2010 Q2 MS

2. The expression of cosh a in exponentials enables the integral to be written as

$$\int_0^1 \frac{1}{x^2 + x(e^a + e^{-a}) + 1} dx$$

which can in turn can be expressed as

$$\int_0^1 \frac{1}{(x + e^a)(x + e^{-a})} dx$$

and so employing partial fractions this is

$$\frac{1}{(e^a - e^{-a})}\left[ln\left(\frac{x + e^{-a}}{x + e^a}\right)\right]_0^1$$

The evaluation of this with simplification of logarithms yields

$$\frac{1}{2\sinh a}\left(ln\left(e^a \frac{1 + e^a}{1 + e^a}\right)\right)$$

giving the required result.
In part (ii), the same technique can be employed for both integrals giving, in the first case

$$\int_1^\infty \frac{1}{(x + e^a)(x - e^{-a})} dx$$

$$= \frac{1}{(e^a + e^{-a})}\left[ln\left(\frac{x - e^{-a}}{x + e^a}\right)\right]_1^\infty$$

$$= \frac{1}{2\cosh a}\left(a + ln\left(\coth\frac{a}{2}\right)\right)$$

and in the second

$$\int_0^\infty \frac{1}{(x^2 + e^a)(x^2 + e^{-a})} dx$$

$$= \frac{1}{(e^a - e^{-a})}\left[\frac{1}{e^{-\frac{a}{2}}}\tan^{-1}\left(\frac{x}{e^{-\frac{a}{2}}}\right) - \frac{1}{e^{\frac{a}{2}}}\tan^{-1}\left(\frac{x}{e^{\frac{a}{2}}}\right)\right]_0^\infty$$

$$= \frac{1}{2\sinh a}\left(\frac{\pi}{2} 2\sinh\frac{a}{2}\right)$$

or alternatively

$$\frac{\pi}{4\cosh\frac{a}{2}}$$