STEP III, 2010 Q1 MS

1. The first two parts are obtained by separating off the final term of the summation and expanding the brackets respectively giving $C = \frac{1}{n+1}(nA + x_{n+1})$, and

$$B = \frac{1}{n}\sum_{k=1}^{n} {x_k}^2 - A^2$$

(the latter given in the question) .
By comparison with the expression for B,

$$D = \frac{1}{n+1}\sum_{k=1}^{n+1} {x_k}^2 - C^2$$

which by substituting for

$$\frac{1}{n}\sum_{k=1}^{n} {x_k}^2$$

from the expression for B gives

$$D = \frac{1}{n+1}[n(B + A^2) + {x_{n+1}}^2] - C^2$$

.
Substituting for C from the initial result, the required expression can be obtained which can most neatly be written

$$D = \frac{n}{(n+1)^2}[(n+1)B + (A - x_{n+1})^2]$$

Thus $(n+1)D = nB + \frac{n}{n+1}(A - x_{n+1})^2$ yielding the first inequality.

Also, $D - B = \frac{n}{(n+1)^2}(A - x_{n+1})^2 - \frac{1}{n+1}B$ and this quadratic expression is only negative if and only if $(A - x_{n+1})^2 < \frac{n+1}{n}B$.
Rearranging the inequality to make x_{n+1} the subject yields the required result.