STEP III, 2010 Q13 MS

13. $Corr(Z_1, Z_2) = 0$

$E(Y_2) = E\left(\rho_{12}Z_1 + (1-\rho_{12}^2)^{\frac{1}{2}}Z_2\right) = \rho_{12}E(Z_1) + (1-\rho_{12}^2)^{\frac{1}{2}}E(Z_2) = 0$

$Var(Y_2) = Var\left(\rho_{12}Z_1 + (1-\rho_{12}^2)^{\frac{1}{2}}Z_2\right) = \rho_{12}^2 Var(Z_1) + (1-\rho_{12}^2)Var(Z_2)$

$= \rho_{12}^2 + (1-\rho_{12}^2) = 1$

As $E(Y_1) = E(Y_2) = 0$ and $Var(Y_1) = Var(Y_2) = 1$,

$Corr(Y_1, Y_2) = \frac{Cov(Y_1,Y_2)}{\sqrt{Var(Y_1)Var(Y_2)}} = Cov(Y_1, Y_2) = E(Y_1Y_2) - E(Y_1)E(Y_2)$

$= E\left(\rho_{12}Z_1^2 + (1-\rho_{12}^2)^{\frac{1}{2}}Z_1Z_2\right) = \rho_{12}Var(Z_1) + (1-\rho_{12}^2)^{\frac{1}{2}}E(Z_1)E(Z_2) = \rho_{12}$

$E(Y_3) = E(aZ_1 + bZ_2 + cZ_3) = aE(Z_1) + bE(Z_2) + cE(Z_3) = 0$ is given.

$Var(Y_3) = 1$ implies $a^2 + b^2 + c^2 = 1$

$Corr(Y_1, Y_3) = \rho_{13}$ implies $a = \rho_{13}$ as seen before.

$Corr(Y_2, Y_3) = \rho_{23}$ implies $\rho_{12}a + (1-\rho_{12}^2)^{\frac{1}{2}}b = \rho_{23}$

and hence $a = \rho_{13}$, $b = \frac{\rho_{23}-\rho_{12}\rho_{13}}{(1-\rho_{12}^2)^{\frac{1}{2}}}$, $c = \sqrt{1 - \rho_{13}^2 - \frac{(\rho_{23}-\rho_{12}\rho_{13})^2}{(1-\rho_{12}^2)}}$

$X_i = \mu_i + \sigma_i Y_i$ for $i = 1,2,3$ as $E(X_i) = E(\mu_i + \sigma_i Y_i) = E(\mu_i) + E(\sigma_i Y_i) = \mu_i + \sigma_i E(Y_i) = \mu_i$,

$Var(X_i) = Var(\mu_i + \sigma_i Y_i) = Var(\sigma_i Y_i) = \sigma_i^2 Var(Y_i) = \sigma_i^2$, and

$Corr(X_i, X_j) = Corr(Y_i, Y_j) = \rho_{ij}$ as a linear transformation does not affect correlation.