

STEP III, 2010 Q13 MS

$$13. \text{Corr}(Z_1, Z_2) = 0$$

$$E(Y_2) = E\left(\rho_{12}Z_1 + (1 - \rho_{12}^2)^{\frac{1}{2}}Z_2\right) = \rho_{12}E(Z_1) + (1 - \rho_{12}^2)^{\frac{1}{2}}E(Z_2) = 0$$

$$\begin{aligned} \text{Var}(Y_2) &= \text{Var}\left(\rho_{12}Z_1 + (1 - \rho_{12}^2)^{\frac{1}{2}}Z_2\right) = \rho_{12}^2\text{Var}(Z_1) + (1 - \rho_{12}^2)\text{Var}(Z_2) \\ &= \rho_{12}^2 + (1 - \rho_{12}^2) = 1 \end{aligned}$$

As $E(Y_1) = E(Y_2) = 0$ and $\text{Var}(Y_1) = \text{Var}(Y_2) = 1$,

$$\text{Corr}(Y_1, Y_2) = \frac{\text{Cov}(Y_1, Y_2)}{\sqrt{\text{Var}(Y_1)\text{Var}(Y_2)}} = \text{Cov}(Y_1, Y_2) = E(Y_1Y_2) - E(Y_1)E(Y_2)$$

$$= E\left(\rho_{12}Z_1^2 + (1 - \rho_{12}^2)^{\frac{1}{2}}Z_1Z_2\right) = \rho_{12}\text{Var}(Z_1) + (1 - \rho_{12}^2)^{\frac{1}{2}}E(Z_1)E(Z_2) = \rho_{12}$$

$E(Y_3) = E(aZ_1 + bZ_2 + cZ_3) = aE(Z_1) + bE(Z_2) + cE(Z_3) = 0$ is given.

$\text{Var}(Y_3) = 1$ implies $a^2 + b^2 + c^2 = 1$

$\text{Corr}(Y_1, Y_3) = \rho_{13}$ implies $a = \rho_{13}$ as seen before.

$$\text{Corr}(Y_2, Y_3) = \rho_{23} \text{ implies } \rho_{12}a + (1 - \rho_{12}^2)^{\frac{1}{2}}b = \rho_{23}$$

$$\text{and hence } a = \rho_{13}, b = \frac{\rho_{23} - \rho_{12}\rho_{13}}{(1 - \rho_{12}^2)^{\frac{1}{2}}}, c = \sqrt{1 - \rho_{13}^2 - \frac{(\rho_{23} - \rho_{12}\rho_{13})^2}{(1 - \rho_{12}^2)}}$$

$X_i = \mu_i + \sigma_i Y_i$ for $i = 1, 2, 3$ as $E(X_i) = E(\mu_i + \sigma_i Y_i) = E(\mu_i) + E(\sigma_i Y_i) = \mu_i + \sigma_i E(Y_i) = \mu_i$,

$\text{Var}(X_i) = \text{Var}(\mu_i + \sigma_i Y_i) = \text{Var}(\sigma_i Y_i) = \sigma_i^2 \text{Var}(Y_i) = \sigma_i^2$, and

$\text{Corr}(X_i, X_j) = \text{Corr}(Y_i, Y_j) = \rho_{ij}$ as a linear transformation does not affect correlation.



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