STEP III, 2010 Q13

13 In this question, $\mathrm{Corr}(U, V)$ denotes the product moment correlation coefficient between the random variables U and V, defined by

$$\mathrm{Corr}(U, V) \equiv \frac{\mathrm{Cov}(U, V)}{\sqrt{\mathrm{Var}\,(U)\mathrm{Var}\,(V)}}\,.$$

The independent random variables Z_1, Z_2 and Z_3 each have expectation 0 and variance 1. What is the value of $\mathrm{Corr}(Z_1, Z_2)$?

Let $Y_1 = Z_1$ and let

$$Y_2 = \rho_{12}Z_1 + (1 - \rho_{12}^2)^{\frac{1}{2}}Z_2\,,$$

where ρ_{12} is a given constant with $-1 < \rho_{12} < 1$. Find $\mathrm{E}(Y_2)$, $\mathrm{Var}\,(Y_2)$ and $\mathrm{Corr}(Y_1, Y_2)$.

Now let $Y_3 = aZ_1 + bZ_2 + cZ_3$, where a, b and c are real constants and $c \geqslant 0$. Given that $\mathrm{E}(Y_3) = 0$, $\mathrm{Var}\,(Y_3) = 1$, $\mathrm{Corr}(Y_1, Y_3) = \rho_{13}$ and $\mathrm{Corr}(Y_2, Y_3) = \rho_{23}$, express a, b and c in terms of ρ_{23}, ρ_{13} and ρ_{12}.

Given constants μ_i and σ_i, for $i = 1$, 2 and 3, give expressions in terms of the Y_i for random variables X_i such that $\mathrm{E}(X_i) = \mu_i$, $\mathrm{Var}\,(X_i) = \sigma_i^2$ and $\mathrm{Corr}(X_i, X_j) = \rho_{ij}$.