STEP III, 2010 Q12 MS

12. $S - rS = 1 + dr + dr^2 + \cdots + dr^n + \cdots$ which is 1 plus an infinite GP. Summing that GP and making S the subject produces the displayed result.

$E(A) = 1a + 2(1-a)a + 3(1-a)^2a + \cdots + n(1-a)^{n-1}a + \cdots$ so making use of the first result with $d = 1, r = (1-a)$, $E(A) = a\left\{\frac{1}{1-(1-a)} + \frac{(1-a)}{(1-(1-a))^2}\right\} = a\left\{\frac{1}{a} + \frac{1-a}{a^2}\right\} = \frac{1}{a}$

$\alpha = a + (1-a)(1-b)\alpha = a + a'b'\alpha$ or alternatively, $\alpha = a + a'b'a + a'^2b'^2a + \cdots$ which both lead to the required result.

$\beta = 1 - \alpha = \frac{a'b}{1-a'b'}$ or alternatively, $\beta = a'b + a'^2b'b + a'^3b'^2b + \cdots = \frac{a'b}{1-a'b'}$

The expected number of shots, S, is given by

$E(S) = 1a + 2a'b + 3a'b'a + 4a'^2b'b + 5a'^2b'^2a + \cdots$

$= a\{1 + 3a'b' + 5a'^2b'^2 + \cdots\} + 2a'b\{1 + 2a'b' + \cdots\}$

which using the initial result of the question $= a\left[\frac{1}{1-a'b'} + \frac{2a'b'}{(1-a'b')^2}\right] + 2a'b\left[\frac{1}{1-a'b'} + \frac{a'b'}{(1-a'b')^2}\right]$ and can be shown to simplify to the required expression.