STEP III, 2010 Q10 MS

10. Resolving perpendicularly to OB, $ma\ddot{\phi} = -T\cos\left(\frac{\pi}{2} - \theta - \phi\right)$, where the tension in the elastic string is $T = \lambda\frac{PB-c}{c}$. The sine rule $\frac{a}{\sin\theta} = \frac{PB}{\sin\phi}$
Putting these three results together gives the required expression.
Also from the sine rule, $\frac{b}{\sin(\theta+\phi)} = \frac{a}{\sin\theta}$, so for ϕ and θ small, $\frac{b}{\theta+\phi} \approx \frac{a}{\theta}$ yielding the desired result.
From this result, θ may be made the subject of the formula, so that the result
$ma\ddot{\phi} = -\lambda\left(\frac{a\sin\phi}{c\sin\theta} - 1\right)\sin(\theta+\phi)$, which for small angles becomes
$ma\ddot{\phi} \approx -\lambda\left(\frac{a\phi}{c\theta} - 1\right)(\theta+\phi)$ can be written $\ddot{\phi} \approx -\frac{\lambda}{ma}\left(\frac{b-a-c}{c}\right)\left(\frac{b}{b-a}\right)\phi$
and hence the period is $\tau \approx 2\pi\sqrt{\frac{mac(b-a)}{\lambda b(b-a-c)}}$.