STEP III, 2010 Q1

1 Let $x_1, x_2, \ldots, x_n$ and x_{n+1} be any fixed real numbers. The numbers A and B are defined by

$$A = \frac{1}{n}\sum_{k=1}^{n} x_k\,, \quad B = \frac{1}{n}\sum_{k=1}^{n}(x_k - A)^2\,,$$

and the numbers C and D are defined by

$$C = \frac{1}{n+1}\sum_{k=1}^{n+1} x_k\,, \quad D = \frac{1}{n+1}\sum_{k=1}^{n+1}(x_k - C)^2\,.$$

(i) Express C in terms of A, x_{n+1} and n.

(ii) Show that $B = \frac{1}{n}\sum_{k=1}^{n} x_k^2 - A^2$.

(iii) Express D in terms of B, A, x_{n+1} and n.
Hence show that $(n+1)D \geqslant nB$ for all values of x_{n+1}, but that $D < B$ if and only if

$$A - \sqrt{\frac{(n+1)B}{n}} < x_{n+1} < A + \sqrt{\frac{(n+1)B}{n}}\,.$$