

STEP III, 2009 Q9 MS

9. (i) With V as the speed of projection from P , x and y the horizontal and vertical displacements from P at a time t after projection, and T the time of flight from P to Q , then

$$x = Vt \cos \theta, \quad y = Vt \sin \theta - \frac{1}{2}gt^2, \quad \dot{x} = V \cos \theta, \quad \text{and} \quad \dot{y} = V \sin \theta - gt$$

$$\text{So } \tan \alpha = \frac{VT \tan \theta - \frac{1}{2}gT^2}{VT \cos \theta} = \tan \theta - \frac{gT}{2V \cos \theta}, \quad \text{and} \quad \tan \varphi = \frac{V \sin \theta - gT}{V \cos \theta} = \tan \theta - \frac{gT}{V \cos \theta}$$

$$\text{Thus } \tan \theta + \tan \varphi = 2 \tan \theta - \frac{gT}{V \cos \theta} = 2 \tan \alpha$$

(ii) Using the trajectory equation written as a quadratic equation in $\tan \theta$,

$$\frac{gx^2}{2V^2} \tan^2 \theta - x \tan \theta + \left(\frac{gx^2}{2V^2} + y \right) = 0, \quad \text{giving } \tan \theta + \tan \theta' = \frac{2V^2}{gx}, \quad \text{and}$$

$$\tan \theta \tan \theta' = 1 + \frac{2V^2 y}{gx^2} = 1 + \frac{2V^2}{gx} \tan \alpha.$$

Applying the compound angle formula and substituting, $\tan(\theta + \theta') = -\cot \alpha$

$$\text{So, } +\theta' = \frac{\pi}{2} + \alpha + n\pi, \quad \text{and as } 0 < \theta < \frac{\pi}{2}, \quad 0 < \theta' < \frac{\pi}{2}, \quad 0 < \alpha < \frac{\pi}{2},$$

$$\theta + \theta' = \frac{\pi}{2} + \alpha.$$

Reversing the motion we have, $(-\varphi) + (-\varphi') = \frac{\pi}{2} + (-\alpha) + n'\pi$, and therefore,

$$\varphi + \varphi' = \alpha + \left(-n' - \frac{1}{2}\right)\pi = \theta + \theta' - n'\pi$$

$$0 < \theta < \frac{\pi}{2}, \quad 0 < \theta' < \frac{\pi}{2}, \quad -\frac{\pi}{2} < \varphi < \frac{\pi}{2}, \quad -\frac{\pi}{2} < \varphi' < \frac{\pi}{2}, \quad \text{and } \varphi < \theta, \quad \varphi' < \theta'$$

$$\text{so } \varphi + \varphi' = \theta + \theta' - \pi, \quad \text{or as required } \theta + \theta' = \varphi + \varphi' + \pi$$



NextStepMaths.com

To view mark schemes, fully worked solutions and examiner's comments, and for more details about tutoring and other services offered, go to

NextStepMaths.com