STEP III, 2009 Q9 MS

9. (i) With V as the speed of projection from P, x and y the horizontal and vertical displacements from P at a time t after projection, and T the time of flight from P to Q, then

$x = Vt\cos\theta$, $y = Vt\sin\theta - \frac{1}{2}gt^2$, $\dot{x} = V\cos\theta$, and $\dot{y} = V\sin\theta - gt$

So $\tan\alpha = \frac{VT\tan\theta - \frac{1}{2}gT^2}{VT\cos\theta} = \tan\theta - \frac{gT}{2V\cos\theta}$, and $\tan\varphi = \frac{V\sin\theta - gT}{V\cos\theta} = \tan\theta - \frac{gT}{V\cos\theta}$

Thus $\tan\theta + \tan\varphi = 2\tan\theta - \frac{gT}{V\cos\theta} = 2\tan\alpha$

(ii) Using the trajectory equation written as a quadratic equation in $\tan\theta$,

$\frac{gx^2}{2V^2}\tan^2\theta - x\tan\theta + \left(\frac{gx^2}{2V^2} + y\right) = 0$, giving $\tan\theta + \tan\theta' = \frac{2V^2}{gx}$, and

$\tan\theta\tan\theta' = 1 + \frac{2V^2y}{gx^2} = 1 + \frac{2V^2}{gx}\tan\alpha$.

Applying the compound angle formula and substituting, $\tan(\theta + \theta') = -\cot\alpha$

So, $+\theta' = \frac{\pi}{2} + \alpha + n\pi$, and as $0 < \theta < \frac{\pi}{2}$, $0 < \theta' < \frac{\pi}{2}$, $0 < \alpha < \frac{\pi}{2}$,

$\theta + \theta' = \frac{\pi}{2} + \alpha$.

Reversing the motion we have, $(-\varphi) + (-\varphi') = \frac{\pi}{2} + (-\alpha) + n'\pi$, and therefore,

$\varphi + \varphi' = \alpha + \left(-n' - \frac{1}{2}\right)\pi = \theta + \theta' - n''\pi$

$0 < \theta < \frac{\pi}{2}$, $0 < \theta' < \frac{\pi}{2}$, $-\frac{\pi}{2} < \varphi < \frac{\pi}{2}$, $-\frac{\pi}{2} < \varphi' < \frac{\pi}{2}$, and $\varphi < \theta$, $\varphi' < \theta'$

so $\varphi + \varphi' = \theta + \theta' - \pi$, or as required $\theta + \theta' = \varphi + \varphi' + \pi$