STEP III, 2009 Q8 MS

8. (i) Letting $x = e^{-t}$,
$\lim_{x\to 0}[x^m(\ln x)^n] = \lim_{t\to\infty}[(e^{-t})^m(-t)^n] = (-1)^n \lim_{x\to 0}[e^{-mt}t^n] = 0$

and so letting $m = n = 1$, $\lim_{x\to 0}[x\ln x] = 0$.
Thus, $\lim_{x\to 0} x^x = \lim_{x\to 0} e^{x\ln x} = e^{\lim_{x\to 0} x\ln x} = e^0 = 1$

(ii) Integrating by parts,

$$I_{n+1} = \int_0^1 x^m (\ln x)^{n+1} dx = \left[\frac{x^{m+1}(\ln x)^{n+1}}{m+1}\right]_0^1 - \int_0^1 \frac{x^{m+1}}{m+1}\frac{(n+1)(\ln x)^n}{x} dx$$

$$= 0 - 0\ (using\ the\ first\ result) - \int_0^1 \frac{n+1}{m+1} x^m(\ln x)^n dx = -\frac{n+1}{m+1} I_n$$

So $I_n = \frac{-n}{m+1} \times \frac{-(n-1)}{m+1} \times \frac{-(n-2)}{m+1} \times \ldots \times \frac{-1}{m+1} I_0 = \frac{(-1)^n n!}{(m+1)^n}\int_0^1 x^m dx$

$= \frac{(-1)^n n!}{(m+1)^{n+1}}$

(iii) $$\int_0^1 x^x dx = \int_0^1 e^{x\ln x} dx = \int_0^1 1 + x\ln x + \frac{x^2(\ln x)^2}{2!} + \ldots\ldots dx$$

$$= 1 + I_1 + \frac{1}{2!}I_2 + \ldots\ldots \quad = 1 - \left(\frac{1}{2}\right)^2 + \left(\frac{1}{3}\right)^3 - \left(\frac{1}{4}\right)^4 + \ldots\ldots \text{ as required.}$$