

STEP III, 2009 Q8 MS

8. (i) Letting $x = e^{-t}$,
 $\lim_{x \rightarrow 0} [x^m (\ln x)^n] = \lim_{t \rightarrow \infty} [(e^{-t})^m (-t)^n] = (-1)^n \lim_{x \rightarrow 0} [e^{-mt} t^n] = 0$

and so letting $m = n = 1$, $\lim_{x \rightarrow 0} [x \ln x] = 0$.
 Thus, $\lim_{x \rightarrow 0} x^x = \lim_{x \rightarrow 0} e^{x \ln x} = e^{\lim_{x \rightarrow 0} x \ln x} = e^0 = 1$

(ii) Integrating by parts,

$$I_{n+1} = \int_0^1 x^m (\ln x)^{n+1} dx = \left[\frac{x^{m+1} (\ln x)^{n+1}}{m+1} \right]_0^1 - \int_0^1 \frac{x^{m+1} (n+1) (\ln x)^n}{m+1} dx$$

$$= 0 - 0 \text{ (using the first result)} - \int_0^1 \frac{n+1}{m+1} x^m (\ln x)^n dx = -\frac{n+1}{m+1} I_n$$

$$\text{So } I_n = \frac{-n}{m+1} \times \frac{-(n-1)}{m+1} \times \frac{-(n-2)}{m+1} \times \dots \times \frac{-1}{m+1} I_0 = \frac{(-1)^n n!}{(m+1)^n} \int_0^1 x^m dx$$

$$= \frac{(-1)^n n!}{(m+1)^{n+1}}$$

(iii) $\int_0^1 x^x dx = \int_0^1 e^{x \ln x} dx = \int_0^1 1 + x \ln x + \frac{x^2 (\ln x)^2}{2!} + \dots dx$

$$= 1 + I_1 + \frac{1}{2!} I_2 + \dots = 1 - \left(\frac{1}{2}\right)^2 + \left(\frac{1}{3}\right)^3 - \left(\frac{1}{4}\right)^4 + \dots \text{ as required.}$$



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