STEP III, 2009 Q7 MS

7. (i) This result is simply obtained using the principle of mathematical induction. The $n = 1$ case can be established merely by obtaining f_1 and f_2 from the definition, and then substituting these along with f_0.

(ii)

$$P_0(x) = (1+x^2)\frac{1}{1+x^2} = 1$$

$$P_1(x) = (1+x^2)^2 \frac{-2x}{(1+x^2)^2} = -2x$$

$$P_2(x) = (1+x^2)^3 \frac{6x^2-2}{(1+x^2)^3} = 6x^2 - 2$$

$$P_{n+1}(x) - (1+x^2)\frac{dP_n(x)}{dx} + 2(n+1)xP_n(x)$$

which differentiating P_n by the product rule and substituting

$$= (1+x^2)^{n+2} f_{n+1}(x) - (1+x^2)\left((1+x^2)^{n+1} f_{n+1}(x) + (n+1)2x(1+x^2)^n f_n(x)\right) + 2(n+1)x(1+x^2)^{n+1} f_n(x)$$

which is zero.

Again using the principle of mathematical induction and the result just obtained, it can be found that $P_{k+1}(x)$ is a polynomial of degree not greater than $k+1$.

Further, assuming that $P_k(x)$ has term of highest degree, $(-1)^k (k+1)!\, x^k$, as $P_{n+1}(x) - (1+x^2)\frac{dP_n(x)}{dx} + 2(n+1)xP_n(x) = 0$, the term of highest degree of $P_{k+1}(x)$ is

$$(-1)^k (k+1)!\, kx^{k-1}x^2 - 2(k+1)x(-1)^k (k+1)!\, x^k$$

$$= (-1)^{k+1}(k+2)!\, x^{k+1}$$ as required.

(The form of the term need not be determined, but it must be shown to be non-zero.)