

STEP III, 2009 Q6 MS

6. Using Euler, $e^{i\beta} - e^{i\alpha} = (\cos \beta - \cos \alpha) + i(\sin \beta - \sin \alpha)$

and so

$$|e^{i\beta} - e^{i\alpha}|^2 = (\cos \beta - \cos \alpha)^2 + (\sin \beta - \sin \alpha)^2$$

which can be expanded, and then using Pythagoras, compound and half angle formulae this becomes

$$4 \sin^2 \frac{1}{2}(\beta - \alpha)$$

$|e^{i\beta} - e^{i\alpha}| = 2 \sin \frac{1}{2}(\beta - \alpha)$ as both expressions are positive.

Alternative methods employ the factor formulae.

$$\begin{aligned} & |e^{i\alpha} - e^{i\beta}| |e^{i\gamma} - e^{i\delta}| + |e^{i\beta} - e^{i\gamma}| |e^{i\alpha} - e^{i\delta}| \\ &= 2 \sin\left(\frac{1}{2}(\alpha - \beta)\right) 2 \sin\left(\frac{1}{2}(\gamma - \delta)\right) + 2 \sin\left(\frac{1}{2}(\beta - \gamma)\right) 2 \sin\left(\frac{1}{2}(\alpha - \delta)\right) \end{aligned}$$

which by use of the factor formulae and cancelling terms may be written

$$2 \left(\cos\left(\frac{1}{2}(\alpha - \beta - \gamma + \delta)\right) - \cos\left(\frac{1}{2}(\beta - \gamma + \alpha - \delta)\right) \right)$$

and then again by factor formulae,

$$2 \sin\left(\frac{1}{2}(\alpha - \gamma)\right) 2 \sin\left(\frac{1}{2}(\beta - \delta)\right)$$

which is

$$|e^{i\alpha} - e^{i\gamma}| |e^{i\beta} - e^{i\delta}| \text{ as required.}$$

Thus, the product of the diagonals of a cyclic quadrilateral is equal to the sum of the products of the opposite pairs of sides (Ptolemy's Theorem).



NextStepMaths.com

To view mark schemes, fully worked solutions and examiner's comments, and for more details about tutoring and other services offered, go to NextStepMaths.com