STEP III, 2009 Q6 MS

6. Using Euler, $e^{i\beta} - e^{i\alpha} = (\cos\beta - \cos\alpha) + i(\sin\beta - \sin\alpha)$

and so

$$\left|e^{i\beta} - e^{i\alpha}\right|^2 = (\cos\beta - \cos\alpha)^2 + (\sin\beta - \sin\alpha)^2$$

which can be expanded, and then using Pythagoras, compound and half angle formulae this becomes

$$4\sin^2\frac{1}{2}(\beta - \alpha)$$

$\left|e^{i\beta} - e^{i\alpha}\right| = 2\sin\frac{1}{2}(\beta - \alpha)$ as both expressions are positive.

Alternative methods employ the factor formulae.

$$\left|e^{i\alpha} - e^{i\beta}\right|\left|e^{i\gamma} - e^{i\delta}\right| + \left|e^{i\beta} - e^{i\gamma}\right|\left|e^{i\alpha} - e^{i\delta}\right|$$

$$= 2\sin\left(\frac{1}{2}(\alpha - \beta)\right)2\sin\left(\frac{1}{2}(\gamma - \delta)\right) + 2\sin\left(\frac{1}{2}(\beta - \gamma)\right)2\sin\left(\frac{1}{2}(\alpha - \delta)\right)$$

which by use of the factor formulae and cancelling terms may be written

$$2\left(\cos\left(\frac{1}{2}(\alpha - \beta - \gamma + \delta)\right) - \cos\left(\frac{1}{2}(\beta - \gamma + \alpha - \delta)\right)\right)$$

and then again by factor formulae,

$$2\sin\left(\frac{1}{2}(\alpha - \gamma)\right)2\sin\left(\frac{1}{2}(\beta - \delta)\right)$$

which is

$\left|e^{i\alpha} - e^{i\gamma}\right|\left|e^{i\beta} - e^{i\delta}\right|$ as required.

Thus, the product of the diagonals of a cyclic quadrilateral is equal to the sum of the products of the opposite pairs of sides (Ptolemy's Theorem).