STEP III, 2009 Q5 MS

5. The first result may be obtained by considering

$$(x+y+z)^2-(x^2+y^2+z^2)=2(yz+zx+xy) \quad ,$$

the second by

$$(x^2+y^2+z^2)(x+y+z)=x^3+y^3+z^3+(x^2y+x^2z+y^2z+y^2z+z^2x+z^2y)$$

and the third by

$$(x+y+z)^3=(x^3+y^3+z^3)+3(x^2y+x^2z+y^2z+y^2z+z^2x+z^2y)+6xyz$$

Considering sums and products of roots, we can deduce that x satisfies the cubic equation $x^3-x^2-\frac{1}{2}x-\frac{1}{6}=0$, as do y and z by symmetry. Multiplying by x^{n-2} , $x^{n+1}=x^n+\frac{1}{2}x^{n-1}+\frac{1}{6}x^{n-2}$, with similar results for y and z. Summing these yields

$$S_{n+1}=S_n+\frac{1}{2}S_{n-1}+\frac{1}{6}S_{n-2}$$

Alternatively,

$x^{n+1}+y^{n+1}+z^{n+1}=(x+y+z)(x^n+y^n+z^n)-(xy^n+xz^n+yx^n+yz^n+zx^n+zy^n)$

$=1.S_n-(xy+yz+zx)(x^{n-1}+y^{n-1}+z^{n-1})+xyz(x^{n-2}+y^{n-2}+z^{n-2})$

to give the result.