

STEP III, 2009 Q5 MS

5. The first result may be obtained by considering

$$(x + y + z)^2 - (x^2 + y^2 + z^2) = 2(yz + zx + xy) \quad ,$$

the second by

$$(x^2 + y^2 + z^2)(x + y + z) = x^3 + y^3 + z^3 + (x^2y + x^2z + y^2z + y^2x + z^2x + z^2y)$$

and the third by

$$(x + y + z)^3 = (x^3 + y^3 + z^3) + 3(x^2y + x^2z + y^2z + y^2x + z^2x + z^2y) + 6xyz$$

Considering sums and products of roots, we can deduce that x satisfies the cubic equation $x^3 - x^2 - \frac{1}{2}x - \frac{1}{6} = 0$, as do y and z by symmetry. Multiplying by x^{n-2} , $x^{n+1} = x^n + \frac{1}{2}x^{n-1} + \frac{1}{6}x^{n-2}$, with similar results for y and z . Summing these yields

$$S_{n+1} = S_n + \frac{1}{2}S_{n-1} + \frac{1}{6}S_{n-2}$$

Alternatively,

$$x^{n+1} + y^{n+1} + z^{n+1} = (x + y + z)(x^n + y^n + z^n) - (xy^n + xz^n + yx^n + yz^n + zx^n + zy^n)$$

$$= 1 \cdot S_n - (xy + yz + zx)(x^{n-1} + y^{n-1} + z^{n-1}) + xyz(x^{n-2} + y^{n-2} + z^{n-2})$$

to give the result.



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