STEP III, 2009, Q5

5 The numbers x, y and z satisfy

$$\begin{aligned} x+y+z &= 1 \\ x^2+y^2+z^2 &= 2 \\ x^3+y^3+z^3 &= 3\,. \end{aligned}$$

Show that

$$yz+zx+xy = -\frac{1}{2}\,.$$

Show also that $x^2y+x^2z+y^2z+y^2x+z^2x+z^2y = -1$, and hence that

$$xyz = \frac{1}{6}\,.$$

Let $S_n = x^n+y^n+z^n$. Use the above results to find numbers a, b and c such that the relation

$$S_{n+1} = aS_n + bS_{n-1} + cS_{n-2}\,,$$

holds for all n.