STEP III, 2009, Q4

4 For any given (suitable) function f, the *Laplace transform* of f is the function F defined by

$$\mathrm{F}(s) = \int_0^\infty \mathrm{e}^{-st}\mathrm{f}(t)\mathrm{d}t \qquad (s > 0)\,.$$

(i) Show that the Laplace transform of $\mathrm{e}^{-bt}\mathrm{f}(t)$, where $b > 0$, is $\mathrm{F}(s+b)$.

(ii) Show that the Laplace transform of $\mathrm{f}(at)$, where $a > 0$, is $a^{-1}\mathrm{F}(\frac{s}{a})$.

(iii) Show that the Laplace transform of $\mathrm{f}'(t)$ is $s\mathrm{F}(s) - \mathrm{f}(0)$.

(iv) In the case $\mathrm{f}(t) = \sin t$, show that $\mathrm{F}(s) = \dfrac{1}{s^2+1}$.

Using only these four results, find the Laplace transform of $\mathrm{e}^{-pt}\cos qt$, where $p > 0$ and $q > 0$.