STEP III, 2009 Q3 MS

3. (i) Substituting the power series and tidying up the algebra yields

$f(t) = \dfrac{1}{\left(1 + \dfrac{t}{2!} + \ldots\right)}$ and so $\lim\limits_{t \to 0} f(t) = 1$.

Similarly, $f'(t) = \dfrac{(e^t - 1) - te^t}{(e^t - 1)^2} = \dfrac{-\dfrac{1}{2} - t\left(\dfrac{1}{2!} - \dfrac{1}{3!}\right) - \ldots}{\left(1 + \dfrac{t}{2!} + \ldots\right)^2}$ and so $\lim\limits_{t \to 0} f'(t) = \dfrac{-1}{2}$

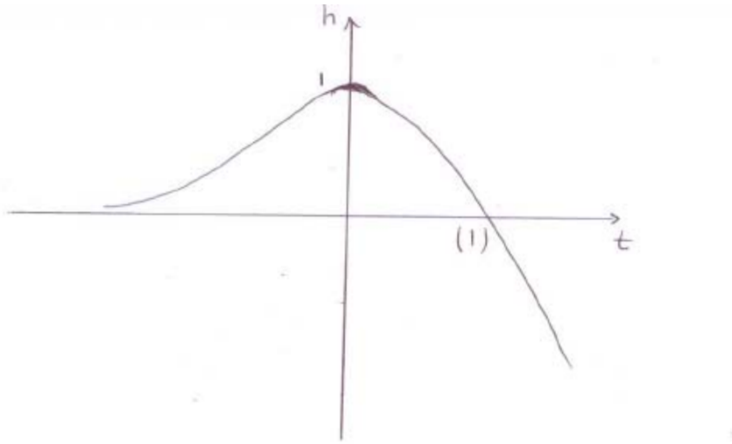
(Alternatively, this can be obtained by de l'Hopital.)

(ii) If we let $g(t) = f(t) + \dfrac{1}{2}t$, then simplifying the algebra gives

$g(t) = \dfrac{t(e^t + 1)}{2(e^t - 1)}$

after which it is can be shown by substituting $-t$ for t that $g(-t)$ is the same expression.

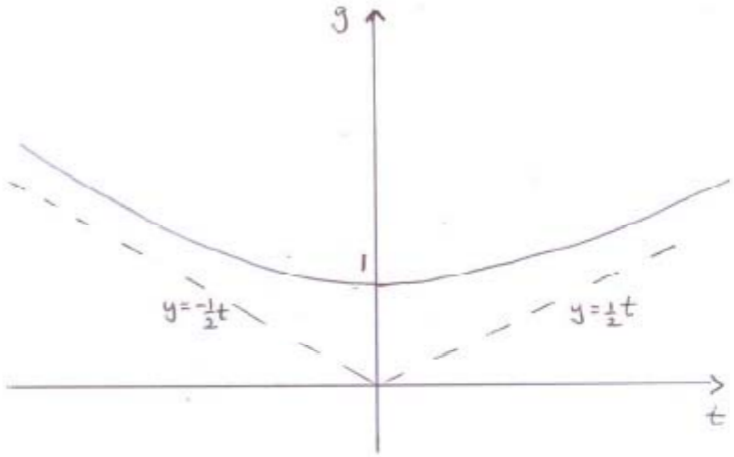
(iii) If we let $h(t) = e^t(1 - t)$, and find its stationary point, sketching the graph gives



Hence $e^t(1 - t) \leq 1$ and so $e^t(1 - t) - 1 \leq 0$. (Alternatively, a sketch with e^t and $\frac{1}{1-t}$ will yield the result.)

Thus $f'(t) = \frac{(1-t)e^t - 1}{(e^t - 1)^2} \leq 0$, with equality only possible for $t = 0$, but we know $\lim\limits_{t \to 0} f'(t) = \dfrac{-1}{2}$ and so, in fact, *f(t)* is always decreasing i.e. has no turning points.

Considering the graph of $g(t) = f(t) + \frac{1}{2}t$. It passes through (0,1), is symmetrical and approaches $y = \frac{1}{2}t$ as $t \to \infty$ and thus is



Therefore the graph of $f(t) = g(t) - \frac{1}{2}t$ also passes through (0,1), and has asymptotes $y = 0$ and $y = -t$ and thus is