

## STEP III, 2009 Q3 MS

3. (i) Substituting the power series and tidying up the algebra yields

$$f(t) = \frac{1}{\left(1 + \frac{t}{2!} + \dots\right)} \text{ and so } \lim_{t \rightarrow 0} f(t) = 1.$$

$$\text{Similarly, } f'(t) = \frac{(e^t - 1) - te^t}{(e^t - 1)^2} = \frac{-\frac{1}{2} - t\left(\frac{1}{2!} - \frac{1}{3!}\right) - \dots}{\left(1 + \frac{t}{2!} + \dots\right)^2} \text{ and so } \lim_{t \rightarrow 0} f'(t) = \frac{-1}{2}$$

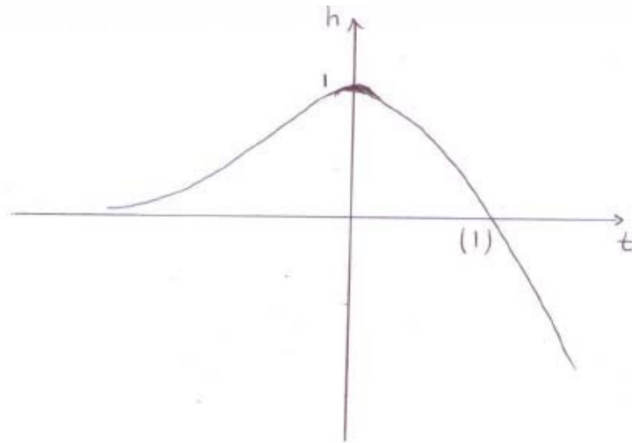
(Alternatively, this can be obtained by de l'Hopital.)

- (ii) If we let  $g(t) = f(t) + \frac{1}{2}t$ , then simplifying the algebra gives

$$g(t) = \frac{t(e^t + 1)}{2(e^t - 1)}$$

after which it is can be shown by substituting  $-t$  for  $t$  that  $g(-t)$  is the same expression.

- (iii) If we let  $h(t) = e^t(1 - t)$ , and find its stationary point, sketching the graph gives



Hence  $e^t(1 - t) \leq 1$  and so  $e^t(1 - t) - 1 \leq 0$ . (Alternatively, a sketch with  $e^t$  and  $\frac{1}{1-t}$  will yield the result.)

Thus  $f'(t) = \frac{(1-t)e^t - 1}{(e^t - 1)^2} \leq 0$ , with equality only possible for  $t = 0$ , but we know

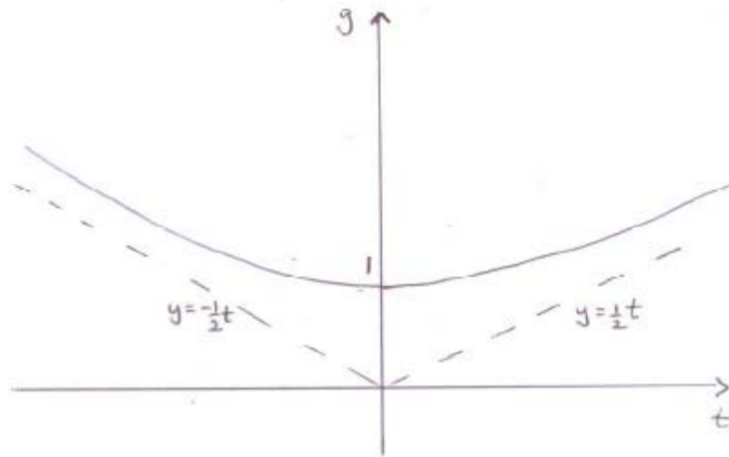
$\lim_{t \rightarrow 0} f'(t) = \frac{-1}{2}$  and so, in fact,  $f(t)$  is always decreasing i.e. has no turning points.



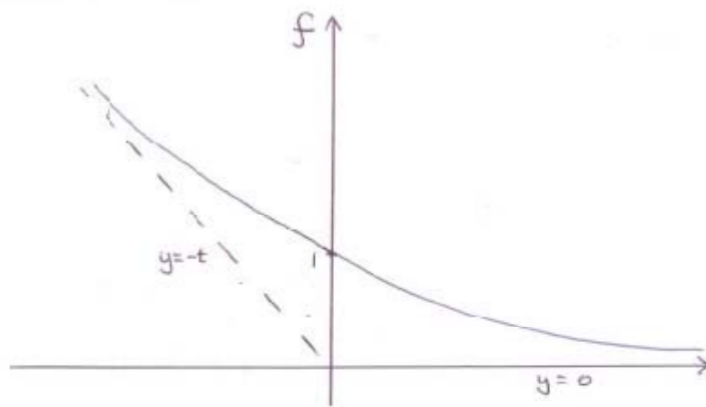
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Considering the graph of  $g(t) = f(t) + \frac{1}{2}t$ . It passes through  $(0,1)$ , is symmetrical and approaches  $y = \frac{1}{2}t$  as  $t \rightarrow \infty$  and thus is



Therefore the graph of  $f(t) = g(t) - \frac{1}{2}t$  also passes through  $(0,1)$ , and has asymptotes  $y = 0$  and  $y = -t$  and thus is



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