STEP III, 2009 Q1 MS

1. The result for p can be found via calculating the equation of the line SV ($y - ms = \dfrac{ms - nv}{s - v}(x - s)$) or similar triangles. The result for q follows from that for p (given in the question) by suitable interchange of letters to give
$q = \dfrac{(m-n)tu}{mt - nu}$

As S and T lie on the circle, s and t are solutions of the equation
$\lambda^2 + (m\lambda - c)^2 = r^2$ i.e. $(1+m^2)\lambda^2 - 2mc\lambda + (c^2 - r^2) = 0$
and so from considering sum and product of roots, $st = \dfrac{c^2 - r^2}{1+m^2}$, and $s + t = \dfrac{2mc}{1+m^2}$
Similarly $uv = \dfrac{c^2 - r^2}{1+n^2}$, and $u + v = \dfrac{2nc}{1+n^2}$ can be deduced by interchanging letters.

Substituting from the earlier results $p + q = \dfrac{(m-n)sv}{ms - nv} + \dfrac{(m-n)tu}{mt - nu}$ which can be simplified to $\dfrac{(m-n)}{(ms - nv)(mt - nu)}\left(stm(u+v) - nuv(s+t)\right)$
and then substituting the sum and product results yields the required result.