

STEP III, 2009 Q13 MS

13.

$$(i) \quad F(x) = P(X < x) = P(\cos \theta < x) = P(\cos^{-1} x < \theta < 2\pi - \cos^{-1} x)$$

$$\text{Therefore, } F(x) = \frac{2\pi - 2\cos^{-1} x}{2\pi}$$

$$\text{So } f(x) = \frac{dF}{dx} = \frac{1}{\pi\sqrt{1-x^2}}, \text{ for } -1 \leq x \leq 1$$

$$E(X) = 0$$

$$E(X^2) = \int_{-1}^1 x^2 \frac{1}{\pi\sqrt{1-x^2}} dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sin^2 u}{\pi} du = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1 - \cos 2u}{2\pi} du = \frac{1}{2}$$

$$\text{So } \text{Var}(X) = \frac{1}{2}$$

If $X = x$, $Y = \pm\sqrt{1-x^2}$ equiprobably, so $E(XY) = 0$, $E(Y) = 0$ and thus $\text{Cov}(X, Y) = 0$, and hence $\text{Corr}(X, Y) = 0$.

X and Y are not independent for if $X = x$, $Y = \pm\sqrt{1-x^2}$ only, whereas without the restriction, Y can take all values in $[-1, 1]$.

$$(ii) \quad E(\bar{X}) = E\left(\frac{1}{n} \sum_{i=1}^n X_i\right) = \frac{1}{n} \sum_{i=1}^n E(X_i) = 0, \text{ and } E(\bar{Y}) = 0 \text{ similarly.}$$

$E(\bar{X}\bar{Y}) = E\left(\frac{1}{n^2} \sum_{i=1}^n X_i \sum_{j=1}^n Y_j\right) = E\left(\frac{1}{n^2} \sum_{i=1}^n X_i Y_i\right)$ as X_i, Y_j are independent and each have expectation zero.

$E\left(\frac{1}{n^2} \sum_{i=1}^n X_i Y_i\right) = 0$ from part (i), and so $E(\bar{X}\bar{Y}) = 0$. Thus $\text{Cov}(\bar{X}, \bar{Y}) = 0$, and hence $\text{Corr}(\bar{X}, \bar{Y}) = 0$ as required.

For large n , $\bar{X} \sim N\left(0, \frac{1}{2n}\right)$ approximately, by Central Limit Theorem.

Thus,

$$P\left(|\bar{X}| \leq \sqrt{\frac{2}{n}}\right) \approx P\left(|z| \leq \frac{\sqrt{\frac{2}{n}}}{\frac{1}{\sqrt{2n}}}\right) = P(|z| \leq 2) \approx P(|z| \leq 1.960) \approx 0.95$$



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