STEP III, 2009 Q13 MS

13.

(i) $F(x) = P(X < x) = P(\cos\theta < x) = P(\cos^{-1}x < \theta < 2\pi - \cos^{-1}x)$

Therefore, $F(x) = \frac{2\pi - 2\cos^{-1}x}{2\pi}$

So $(x) = \frac{dF}{dx} = \frac{1}{\pi\sqrt{1-x^2}}$, for $-1 \le x \le 1$

$E(X) = 0$

$E(X^2) = \int_{-1}^{1} x^2 \frac{1}{\pi\sqrt{1-x^2}} \quad dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sin^2 u}{\pi} du = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1-\cos 2u}{2\pi} du = \frac{1}{2}$

So $Var(X) = \frac{1}{2}$

If $X = x$, $Y = \pm\sqrt{1-x^2}$ equiprobably, so $E(XY) = 0$, $E(Y) = 0$ and thus $Cov(X,Y) = 0$, and hence $Corr(X,Y) = 0$.

X and Y are not independent for if $X = x$, $Y = \pm\sqrt{1-x^2}$ only, whereas without the restriction, Y can take all values in $[-1,1]$.

(ii) $E(\bar{X}) = E\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n}\sum_{i=1}^{n} E(X_i) = 0$, and $E(\bar{Y}) = 0$ similarly.

$E(\bar{X}\bar{Y}) = E\left(\frac{1}{n^2}\sum_{i=1}^{n} X_i \sum_{j=i}^{n} Y_j\right) = E\left(\frac{1}{n^2}\sum_{i=1}^{n} X_i Y_i\right)$ as X_i, Y_j are independent and each have expectation zero.

$E\left(\frac{1}{n^2}\sum_{i=1}^{n} X_i Y_i\right) = 0$ from part (i), and so $E(\bar{X}\bar{Y}) = 0$. Thus $Cov(\bar{X},\bar{Y}) = 0$, and hence $Corr(\bar{X},\bar{Y}) = 0$ as required.

For large n , $\bar{X} \sim N\left(0, \frac{1}{2n}\right)$ approximately, by Central Limit Theorem.

Thus,

$$P\left(|\bar{X}| \le \sqrt{\frac{2}{n}}\right) \approx P\left(|z| \le \frac{\sqrt{\frac{2}{n}}}{\frac{1}{\sqrt{2n}}}\right) = P(|z| \le 2) \approx P(|z| \le 1\cdot 960) \approx 0\cdot 95$$