

## STEP III, 2009 Q12 MS

12. (i)  $E(X_1) = \frac{1}{2}k$ ,  $E(X_2|X_1 = x_1) = \frac{1}{2}x_1$ , and so  
 $E(X_2) = \sum \frac{1}{2}x_1 P(X_1 = x_1) = \frac{1}{2}E(X_1) = \frac{1}{4}k$   
 $\sum_{i=1}^{\infty} E(X_i) = \sum_{i=1}^{\infty} \left(\frac{1}{2}\right)^i k = k$  using the sum of an infinite GP.

(ii)  $G_Y(t) = E(t^Y) = E\left(t^{\sum_{i=1}^k Y_i}\right) = \prod_{i=1}^k E(t^{Y_i})$   
 $P(Y_i = 0) = \frac{1}{2}$ ,  $P(Y_i = 1) = \frac{1}{4}$ , ...,  $P(Y_i = r) = \left(\frac{1}{2}\right)^{r-1}$   
and so  $E(t^{Y_i}) = \frac{1}{2} + \frac{1}{4}t + \dots + \left(\frac{1}{2}\right)^{r-1} t^r + \dots = \frac{\frac{1}{2}}{\left(1 - \frac{1}{2}t\right)} = \frac{1}{2-t}$  (infinite GP)

Thus  $G_Y(t) = \prod_{i=1}^k \frac{1}{2-t} = \left(\frac{1}{2-t}\right)^k$

$G'_Y(t) = \frac{k}{(2-t)^{k+1}}$ ,  $G''_Y(t) = \frac{k(k+1)}{(2-t)^{k+2}}$ , and  $G^{(r)}_Y(t) = \frac{k(k+1)(k+2)\dots(k+r-1)}{(2-t)^{k+r}}$   
and so  $E(Y) = G'_Y(1) = k$ ,  $Var(Y) = G''_Y(1) + G'_Y(1) - \left(G'_Y(1)\right)^2 = 2k$   
and  $P(Y = r) = \frac{G^{(r)}_Y(0)}{r!} = \frac{k(k+1)(k+2)\dots(k+r-1)}{2^{k+r}r!} = {}^{k+r-1}C_r \left(\frac{1}{2}\right)^{k+r}$  for  $r = 0, 1, 2, \dots$

(Alternatively,  $P(Y = r)$  is coefficient of  $t^r$  in  $G_Y(t)$  which can be expanded binomially to yield the same result.)



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