STEP III, 2009 Q12 MS

12. (i) $E(X_1) = \frac{1}{2}k$, $E(X_2|X_1 = x_1) = \frac{1}{2}x_1$, and so
$E(X_2) = \sum \frac{1}{2}x_1 P(X_1 = x_1) = \frac{1}{2}E(X_1) = \frac{1}{4}k$
$\sum_{i=1}^{\infty} E(X_i) = \sum_{i=1}^{\infty} \left(\frac{1}{2}\right)^i k = k$ using the sum of an infinite GP.

(ii) $G_Y(t) = E(t^Y) = E\left(t^{\sum_{i=1}^{k} Y_i}\right) = \prod_{i=1}^{k} E(t^{Y_i})$
$P(Y_i = 0) = \frac{1}{2}$, $(Y_i = 1) = \frac{1}{4}$, … , $P(Y_i = r) = \left(\frac{1}{2}\right)^{r-1}$
and so $E(t^{Y_i}) = \frac{1}{2} + \frac{1}{4}t + \cdots + \left(\frac{1}{2}\right)^{r-1} t^r + \ \ldots = \frac{\frac{1}{2}}{\left(1-\frac{1}{2}t\right)} = \frac{1}{2-t}$ (infinite GP)
Thus $G_Y(t) = \prod_{i=1}^{k} \frac{1}{2-t} = \left(\frac{1}{2-t}\right)^k$

$G'_Y(t) = \frac{k}{(2-t)^{k+1}}$, $G''_Y(t) = \frac{k(k+1)}{(2-t)^{k+2}}$, and $G^{(r)}{}_Y(t) = \frac{k(k+1)(k+2)\ldots(k+r-1)}{(2-t)^{k+r}}$
and so $E(Y) = G'_Y(1) = k$, $Var(Y) = G''_Y(1) + G'_Y(1) - \left(G'_Y(1)\right)^2 = 2k$
and $P(Y = r) = \frac{G^{(r)}{}_Y(0)}{r!} = \frac{k(k+1)(k+2)\ldots(k+r-1)}{2^{k+r}r!} = {}^{k+r-1}C_r \left(\frac{1}{2}\right)^{k+r}$ for $r = 0, 1, 2, \ldots$

(Alternatively, $P(Y = r)$ is coefficient of t^r in $G_Y(t)$ which can be expanded binomially to yield the same result.)