STEP III 2009 Question 11 — Mark Scheme

(i) Conserving momentum $MV = M(1+bx)v$ and so $V = (1+bx)v$ **M1A1**

$V = (1+bx)\dfrac{dx}{dt}$

$\int V\,dt = \int (1+bx)\,dx$ **M1**

$Vt + c = x + \frac{1}{2}bx^2$, **A1**

and as $= 0$, when $t = 0$, $c = 0$

So $\frac{1}{2}bx^2 + x - Vt = 0$, and so $x = \dfrac{-1 \pm \sqrt{1+2bVt}}{b}$, **M1**

except $x > 0$, and thus $x = \dfrac{-1+\sqrt{1+2bVt}}{b}$ **M1 A1**

7 marks

(ii) $Mf = \dfrac{d}{dt}(mv) = \dfrac{d}{dt}\big(M(1+bx)v\big)$ **B1**

Therefore, $ft + c = (1+bx)v$ **M1 A1**

When $= 0$, $x = 0$, and $v = V$ so $c = V$ **M1**

Thus $v = \dfrac{ft+V}{1+bx}$ as required (*) **A1**

5 marks

$ft + V = (1+bx)\dfrac{dx}{dt}$

$\int ft + V\,dt = \int (1+bx)\,dx$ **M1**

$\frac{1}{2}ft^2 + Vt + c' = x + \frac{1}{2}bx^2$ and as $x = 0$, when $t = 0$, $c' = 0$ **M1**

So $\frac{1}{2}bx^2 + x - \frac{1}{2}ft^2 - Vt = 0$, and so $= \dfrac{-1 \pm \sqrt{1+fbt^2+2bVt}}{b}$, except $x > 0$,

and thus $x = \dfrac{-1+\sqrt{1+fbt^2+2bVt}}{b}$ **M1 A1**

4 marks

If $1 + fbt^2 + 2bVt$ is a perfect square, **M1**

then x will be linear in t and $\dfrac{dx}{dt}$ will be constant, i.e. if $4b^2V^2 - 4fb = 0$, that is

$bV^2 = f$ **A1**

(in which case $x = \dfrac{-1+\sqrt{1+b^2V^2t^2+2bVt}}{b} = \dfrac{-1+(1+bVt)}{b} = Vt$, and $v = V$ as expected.)

Otherwise, $= \dfrac{ft+V}{1+bx} = \dfrac{ft+V}{\sqrt{1+fbt^2+2bVt}} = \dfrac{f+\frac{V}{t}}{\sqrt{fb+\frac{2bV}{t}+\frac{1}{t^2}}}$, and as $t \to \infty$, $v \to \dfrac{f}{\sqrt{fb}} = \sqrt{\dfrac{f}{b}}$

a constant, as required. **M1 A1**

4 marks