STEP III, 2009 Q11 MS

11. (i) Conserving momentum yields $MV = M(1+bx)v$ and so $V = (1+bx)v$ Written as $V = (1+bx)\frac{dx}{dt}$, separating variables and integrating $Vt + c = x + \frac{1}{2}bx^2$, but as $= 0$, when $t = 0$, $c = 0$

So $\frac{1}{2}bx^2 + x - Vt = 0$, and so $x = \frac{-1\pm\sqrt{1+2bVt}}{b}$, except $x > 0$, and thus $x = \frac{-1+\sqrt{1+2bVt}}{b}$

(ii) $Mf = \frac{d}{dt}(mv) = \frac{d}{dt}(M(1+bx)v)$

So, $ft + c' = (1+bx)v$ and as $= 0$, $x = 0$, and $v = V$ we have $c' = V$.

Thus $v = \frac{ft+V}{1+bx}$ as required.

Separating variables and integrating
$\frac{1}{2}ft^2 + Vt + c'' = x + \frac{1}{2}bx^2$ and as $x = 0$, when $t = 0$, $c'' = 0$
So $\frac{1}{2}bx^2 + x - \frac{1}{2}ft^2 - Vt = 0$, and so $= \frac{-1\pm\sqrt{1+fbt^2+2bVt}}{b}$, except $x > 0$, and thus $x = \frac{-1+\sqrt{1+fbt^2+2bVt}}{b}$

If $1 + fbt^2 + 2bVt$ is a perfect square, then x will be linear in t and $\frac{dx}{dt}$ will be constant, i.e. if $4b^2V^2 - 4fb = 0$, that is $bV^2 = f$
(in which case $x = \frac{-1+\sqrt{1+b^2V^2t^2+2bVt}}{b} = \frac{-1+(1+bVt)}{b} = Vt$, and $v = V$ as expected.)

Otherwise, $= \frac{ft+V}{1+bx} = \frac{ft+V}{\sqrt{1+fbt^2+2bVt}} = \frac{f+\frac{V}{t}}{\sqrt{fb+\frac{2bV}{t}+\frac{1}{t^2}}}$, and as $t \to \infty$, $v \to \frac{f}{\sqrt{fb}} = \sqrt{\frac{f}{b}}$, a constant, as required.