STEP III, 2009, Q11

11 A comet in deep space picks up mass as it travels through a large stationary dust cloud. It is subject to a gravitational force of magnitude Mf acting in the direction of its motion. When it entered the cloud, the comet had mass M and speed V. After a time t, it has travelled a distance x through the cloud, its mass is $M(1+bx)$, where b is a positive constant, and its speed is v.

(i) In the case when $f = 0$, write down an equation relating V, x, v and b. Hence find an expression for x in terms of b, V and t.

(ii) In the case when f is a non-zero constant, use Newton's second law in the form

$$\text{force} = \text{rate of change of momentum}$$

to show that

$$v = \frac{ft+V}{1+bx}\,.$$

Hence find an expression for x in terms of b, V, f and t.

Show that it is possible, if b, V and f are suitably chosen, for the comet to move with constant speed. Show also that, if the comet does not move with constant speed, its speed tends to a constant as $t \to \infty$.