STEP III 2009 Question 10 — Mark Scheme

Supposing that the particle P has mass m, that the spring has natural length l, and modulus of elasticity λ, $mg = \dfrac{\lambda d}{l}$ **M1 A1**

Conserving energy, if the speed of P when it hits the top of the spring is v, $mgh = \frac{1}{2}mv^2$, and so $v = \sqrt{2gh}$ **M1 A1**

Using Newton's second law, the second-order differential equation is thus **M1**

$m\ddot{x} = mg - \dfrac{\lambda x}{l} = mg - \dfrac{mgx}{d}$ and so $\ddot{x} = g - \dfrac{gx}{d}$ **A1**

with initial conditions that $x = 0$, $\dot{x} = \sqrt{2gh}$, when $t = 0$. **B1**

7 marks

$\ddot{x} = g - \dfrac{gx}{d}$, i.e. $\ddot{x} + \dfrac{gx}{d} = g$ has complementary function $x = B\cos\omega t + C\sin\omega t$ **M1**

where $\omega = \sqrt{\dfrac{g}{d}}$, and particular integral $x = A$, where $\dfrac{gA}{d} = g$, i.e. $A = d$.

A1 **M1** **A1**

Using the initial conditions, $0 = d + B$, i.e. $B = -d$ and $\sqrt{2gh} = C\omega$, i.e. $C = \sqrt{2dh}$ **B1** for B, **B1** for C

So $x = d - d\cos\sqrt{\dfrac{g}{d}}t + \sqrt{2dh}\sin\sqrt{\dfrac{g}{d}}t$

6 marks

$d\cos\sqrt{\dfrac{g}{d}}t - \sqrt{2dh}\sin\sqrt{\dfrac{g}{d}}t$ may be expressed in the form $R\cos\left(\sqrt{\dfrac{g}{d}}t + \alpha\right)$ **M1**

where $R^2 = d^2 + 2dh$, and $\tan\alpha = \dfrac{\sqrt{2dh}}{d} = \sqrt{\dfrac{2h}{d}}$ **A1, A1**

Thus $x = d - R\cos\left(\sqrt{\dfrac{g}{d}}t + \alpha\right)$ and $x = 0$ next when $= T$. **M1**

That is when $2\pi - \left(\sqrt{\dfrac{g}{d}}T + \alpha\right) = \alpha$ **M1**

So $\sqrt{\dfrac{g}{d}}T = 2\pi - 2\alpha = 2\pi - 2\tan^{-1}\sqrt{\dfrac{2h}{d}}$, **M1**

and so $T = \sqrt{\dfrac{d}{g}}\left(2\pi - 2\tan^{-1}\sqrt{\dfrac{2h}{d}}\right)$ as required. (*) **A1**

7 marks