STEP III, 2009 Q10 MS

10. Supposing that the particle P has mass m, the spring has natural length l, and modulus of elasticity λ, $mg = \frac{\lambda d}{l}$

If the speed of P when it hits the top of the spring is v, then $v = \sqrt{2gh}$

By Newton's second law, the second-order differential equation is thus

$m\ddot{x} = mg - \frac{\lambda x}{l} = mg - \frac{mgx}{d}$ and so $\ddot{x} = g - \frac{gx}{d}$ with initial conditions that $x = 0$, $\dot{x} = \sqrt{2gh}$, when $t = 0$.

$\ddot{x} + \frac{gx}{d} = g$ has complementary function $x = B\cos\omega t + C\sin\omega t$

where $\omega = \sqrt{\frac{g}{d}}$, and particular integral $x = A$, where $A = d$.

The initial conditions yield, $B = -d$ and $C = \sqrt{2dh}$

So $x = d - d\cos\sqrt{\frac{g}{d}}t + \sqrt{2dh}\sin\sqrt{\frac{g}{d}}t$.

$d\cos\sqrt{\frac{g}{d}}t - \sqrt{2dh}\sin\sqrt{\frac{g}{d}}t$ may be expressed in the form $R\cos\left(\sqrt{\frac{g}{d}}t + \alpha\right)$ where

$R^2 = d^2 + 2dh$, and $\tan\alpha = \frac{\sqrt{2dh}}{d} = \sqrt{\frac{2h}{d}}$

So $x = d - R\cos\left(\sqrt{\frac{g}{d}}t + \alpha\right)$

$x = 0$ next when $t = T$, that is when $2\pi - \left(\sqrt{\frac{g}{d}}T + \alpha\right) = \alpha$

So $\sqrt{\frac{g}{d}}T = 2\pi - 2\alpha = 2\pi - 2\tan^{-1}\sqrt{\frac{2h}{d}}$ and $T = \sqrt{\frac{d}{g}}\left(2\pi - 2\tan^{-1}\sqrt{\frac{2h}{d}}\right)$.