STEP III, 2009, Q10

10 A light spring is fixed at its lower end and its axis is vertical. When a certain particle P rests on the top of the spring, the compression is d. When, instead, P is dropped onto the top of the spring from a height h above it, the compression at time t after P hits the top of the spring is x. Obtain a second-order differential equation relating x and t for $0 \leqslant t \leqslant T$, where T is the time at which P first loses contact with the spring.

Find the solution of this equation in the form

$$x = A + B\cos(\omega t) + C\sin(\omega t)\,,$$

where the constants A, B, C and ω are to be given in terms of d, g and h as appropriate.

Show that

$$T = \sqrt{d/g}\left(2\pi - 2\arctan\sqrt{2h/d}\right)\,.$$