STEP III, 2008 Q8 MS

8. (i) $p = -\frac{1}{2}$

$(1+px)S = \frac{1}{3}x$ with all other terms cancelling and so $S = \frac{1}{3}x \Big/ \left(1 - \frac{1}{2}x\right) = \frac{2x}{3(2-x)}$

Using the sum of a GP

$$S_{n+1} = \frac{1}{3}x + \frac{1}{6}x^2 + \frac{1}{12}x^3 + \ldots + \frac{1}{3\times 2^n}x^{n+1} = \frac{\frac{1}{3}x\left(1 - \frac{x^{n+1}}{2^{n+1}}\right)}{\left(1-\frac{x}{2}\right)}$$

Alternatively $S_{n+1} = S - (a_{n+2}x^{n+2} + \ldots) = S - \frac{1}{2^{n+1}}x^{n+1}S$

$$= \left(1 - \frac{x^{n+1}}{2^{n+1}}\right)\frac{2x}{3(2-x)}$$

(ii) Using similar working to part (i)

$18 + 8p + 2q = 0$

$37 + 18p + 8q = 0$

so $p = -\frac{5}{2}, q = 1$

and so $(1 + px + qx^2)T = 2 + 3x$

giving $T = (2+3x) \Big/ \left(1 - \frac{5}{2}x + x^2\right) = \frac{4+6x}{2-5x+2x^2} = \frac{4+6x}{(2-x)(1-2x)}$

By partial fractions $T = \frac{14}{3}(1-2x)^{-1} - \frac{8}{3}\left(1 - \frac{x}{2}\right)^{-1}$

and so $T_{n+1} = \frac{14}{3}\left(1 + 2x + (2x)^2 + \ldots + (2x)^n\right) - \frac{8}{3}\left(1 + \frac{x}{2} + \left(\frac{x}{2}\right)^2 + \ldots + \left(\frac{x}{2}\right)^n\right)$

$$= \frac{14}{3}\frac{\left(1-(2x)^{n+1}\right)}{1-2x} - \frac{8}{3}\frac{\left(1 - \left(\frac{x}{2}\right)^{n+1}\right)}{1-\frac{x}{2}}$$