STEP III, 2008 Q5 MS

5. There are a number of correct routes to proving the induction, though the simplest is to consider $\left(\left(T_{k+1}(x)\right)^2 - T_k(x)T_{k+2}(x)\right) - \left(\left(T_k(x)\right)^2 - T_{k-1}(x)T_{k+1}(x)\right)$

For $f(x) = 0$, $\left(T_n(x)\right)^2 - T_{n-1}(x)T_{n+1}(x) = 0$

and so $\dfrac{T_{n+1}(x)}{T_n(x)} = \dfrac{T_n(x)}{T_{n-1}(x)}$ provided that neither denominator is zero, leading to

$\dfrac{T_n(x)}{T_{n-1}(x)} = \dfrac{T_1(x)}{T_0(x)} = r(x)$,

and so $\dfrac{T_n(x)}{T_{n-1}(x)} \times \dfrac{T_{n-1}(x)}{T_{n-2}(x)} \times \ldots \times \dfrac{T_1(x)}{T_0(x)} = \left(r(x)\right)^n$

Thus $T_n(x) = \left(r(x)\right)^n T_0(x)$

Substituting this result into (*) for $n = 1$,

$\left(\left(r(x)\right)^2 - 2xr(x) + 1\right)T_0(x) = 0$, and as $T_0(x) \neq 0$, solving the quadratic gives

$r(x) = x \pm \sqrt{x^2 - 1}$