

STEP III, 2008 Q5 MS

5. There are a number of correct routes to proving the induction, though the simplest is to consider $\left((T_{k+1}(x))^2 - T_k(x)T_{k+2}(x)\right) - \left((T_k(x))^2 - T_{k-1}(x)T_{k+1}(x)\right)$

For $f(x) = 0$, $(T_n(x))^2 - T_{n-1}(x)T_{n+1}(x) = 0$

and so $\frac{T_{n+1}(x)}{T_n(x)} = \frac{T_n(x)}{T_{n-1}(x)}$ provided that neither denominator is zero, leading to

$$\frac{T_n(x)}{T_{n-1}(x)} = \frac{T_1(x)}{T_0(x)} = r(x),$$

and so $\frac{T_n(x)}{T_{n-1}(x)} \times \frac{T_{n-1}(x)}{T_{n-2}(x)} \times \dots \times \frac{T_1(x)}{T_0(x)} = (r(x))^n$

Thus $T_n(x) = (r(x))^n T_0(x)$

Substituting this result into (*) for $n = 1$,

$\left((r(x))^2 - 2xr(x) + 1\right)T_0(x) = 0$, and as $T_0(x) \neq 0$, solving the quadratic gives

$$r(x) = x \pm \sqrt{x^2 - 1}$$



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