

STEP III, 2008 Q4 MS

4. (i)



The graph of $z = y$ has gradient 1 and passes through the origin.

The graph of $z = \tanh\left(\frac{y}{2}\right)$ which has gradient $\frac{1}{2} \operatorname{sech}^2\left(\frac{y}{2}\right) \leq \frac{1}{2}$ for $y \geq 0$ also passes through the origin and is asymptotic to $z = 1$.

Thus $y \geq \tanh\left(\frac{y}{2}\right)$ for $y \geq 0$.

$$\text{If } x = \cosh y, \text{ then } \sqrt{\frac{x-1}{x+1}} = \sqrt{\frac{\cosh y - 1}{\cosh y + 1}} = \sqrt{\frac{2 \sinh^2\left(\frac{y}{2}\right)}{2 \cosh^2\left(\frac{y}{2}\right)}} = \tanh\left(\frac{y}{2}\right)$$

and as $y \geq \tanh\left(\frac{y}{2}\right)$ for $y \geq 0$, $\operatorname{ar} \cosh x \geq \sqrt{\frac{x-1}{x+1}}$ for $x \geq 1$.

$$\sqrt{\frac{x-1}{x+1}} = \sqrt{\frac{x-1}{x+1}} \sqrt{\frac{x-1}{x-1}} = \frac{x-1}{\sqrt{x^2-1}} \text{ for } x > 1, \text{ and (*) is obtained.}$$

(ii) By parts $\int \operatorname{ar} \cosh x dx = x \operatorname{ar} \cosh x - \sqrt{x^2-1} + c$

$$\text{and } \int \frac{x-1}{\sqrt{x^2-1}} dx = \sqrt{x^2-1} - \operatorname{ar} \cosh x + c'$$

Thus $\int_1^x \operatorname{ar} \cosh x dx \geq \int_1^x \frac{x-1}{\sqrt{x^2-1}} dx$ for $x > 1$ gives

$$x \operatorname{ar} \cosh x - \sqrt{x^2-1} \geq \sqrt{x^2-1} - \operatorname{ar} \cosh x \text{ for } x > 1, \text{ which rearranges to give result}$$

(iii) Integrating (ii) similarly gives $x \operatorname{ar} \cosh x - \sqrt{x^2-1} \geq 2(\sqrt{x^2-1} - \operatorname{ar} \cosh x)$ for $x > 1$, which also can be rearranged as desired.



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