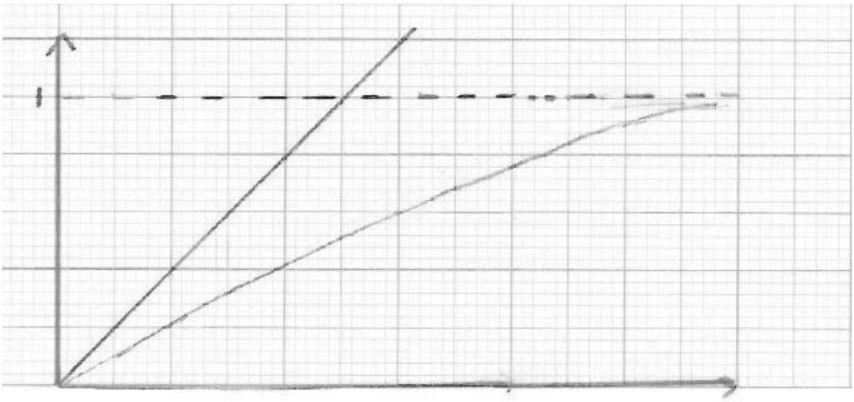
STEP III, 2008 Q4 MS

4. (i)



The graph of $z = y$ has gradient 1 and passes through the origin.

The graph of $z = \tanh\left(\frac{y}{2}\right)$ which has gradient $\frac{1}{2}\operatorname{sech}^2\left(\frac{y}{2}\right) \leq \frac{1}{2}$ for $y \geq 0$ also passes though the origin and is asymptotic to $z = 1$.

Thus $y \geq \tanh\left(\frac{y}{2}\right)$ for $y \geq 0$.

If $x = \cosh y$, then $\sqrt{\frac{x-1}{x+1}} = \sqrt{\frac{\cosh y - 1}{\cosh y + 1}} = \sqrt{\frac{2\sinh^2\left(\frac{y}{2}\right)}{2\cosh^2\left(\frac{y}{2}\right)}} = \tanh\left(\frac{y}{2}\right)$

and as $y \geq \tanh\left(\frac{y}{2}\right)$ for $y \geq 0$, $\operatorname{ar}\cosh x \geq \sqrt{\frac{x-1}{x+1}}$ for $x \geq 1$.

$\sqrt{\frac{x-1}{x+1}} = \sqrt{\frac{x-1}{x+1}}\sqrt{\frac{x-1}{x-1}} = \frac{x-1}{\sqrt{x^2-1}}$ for $x > 1$, and (*) is obtained.

(ii) By parts $\int \operatorname{ar}\cosh x\,dx = x\operatorname{ar}\cosh x - \sqrt{x^2-1} + c$

and $\int \frac{x-1}{\sqrt{x^2-1}}\,dx = \sqrt{x^2-1} - \operatorname{ar}\cosh x + c'$

Thus $\int_1^x \operatorname{ar}\cosh x\,dx \geq \int_1^x \frac{x-1}{\sqrt{x^2-1}}\,dx$ for $x > 1$ gives

$x\operatorname{ar}\cosh x - \sqrt{x^2-1} \geq \sqrt{x^2-1} - \operatorname{ar}\cosh x$ for $x > 1$, which rearranges to give result

(iii) Integrating (ii) similarly gives $x\operatorname{ar}\cosh x - \sqrt{x^2-1} \geq 2\left(\sqrt{x^2-1} - \operatorname{ar}\cosh x\right)$ for $x > 1$, which also can be rearranged as desired.