STEP III, 2008 Q2 MS

2. (i) On the one hand

$\sum_{r=0}^{n}\left[(r+1)^k - r^k\right] = \sum_{r=0}^{n}(r+1)^k - \sum_{r=0}^{n} r^k = \sum_{r=1}^{n+1} r^k - \sum_{r=0}^{n} r^k = (n+1)^k$ whilst expanding binomially yields

$$k\sum_{r=0}^{n} r^{k-1} + \binom{k}{2}\sum_{r=0}^{n} r^{k-2} + \binom{k}{3}\sum_{r=0}^{n} r^{k-3} + \ldots + \binom{k}{k-1}\sum_{r=0}^{n} r^{k-1} + \sum_{r=0}^{n} 1$$

$$= kS_{k-1}(n) + \binom{k}{2}S_{k-2}(n) + \binom{k}{3}S_{k-3}(n) + \ldots + \binom{k}{k-1}S_1(n) + (n+1)$$

and hence the required result.

Applying this in the case $k = 4$ gives

$$4S_3(n) = (n+1)^4 - (n+1) - \binom{4}{2}S_2(n) - \binom{4}{3}S_1(n)$$

which, after substitution of the two given results and factorization, yields the familiar

$$S_3(n) = \frac{1}{4}n^2(n+1)^2$$

The identical process with $k = 5$ results in

$$S_4(n) = \frac{1}{30}n(n+1)\left(6n^3 + 9n^2 + n - 1\right) = \frac{1}{30}n(n+1)(2n+1)\left(3n^2 + 3n - 1\right)$$

(ii) Applying induction, with the assumption that $S_t(n)$ is a polynomial of degree $t+1$ in n for $t < r$ for some r, and then considering (*),

$(n+1)^{r+1} - (n+1)$ is a polynomial of degree $r+1$ in n,

and each of the terms $-\binom{r+1}{j}S_{r+1-j}(n)$ is a polynomial of degree $r+2-j$ in n where $j \geq 2$, i.e. the degree is $\leq r$. A sum of polynomials of degree $\leq r+1$ in n, is a polynomial of degree $\leq r+1$ in n, and there is a single non-zero term in n^{r+1} from just $(n+1)^{r+1}$ so the degree of the polynomial is not reduced to $< r+1$, i.e. it is $r+1$. (The initial case is true to complete the proof.)

If, $S_k(n) = \sum_{i=0}^{k+1} a_i n^i = \sum_{r=0}^{n} r^k$ then $S_k(0) = a_0 + \sum_{i=1}^{k+1} a_i 0^i = \sum_{r=0}^{0} r^k = 0$ and so $a_0 = 0$

$S_k(1) = \sum_{i=0}^{k+1} a_i 1^i = \sum_{r=0}^{1} r^k = 1$ and so $\sum_{i=0}^{k+1} a_i = 1$ as required.