STEP III, 2008 Q2

2 Let $S_k(n) \equiv \sum_{r=0}^{n} r^k$, where k is a positive integer, so that

$$S_1(n) \equiv \tfrac{1}{2}n(n+1) \qquad \text{and} \qquad S_2(n) \equiv \tfrac{1}{6}n(n+1)(2n+1)\,.$$

(i) By considering $\sum_{r=0}^{n} \left[(r+1)^k - r^k\right]$, show that

$$kS_{k-1}(n) = (n+1)^k - (n+1) - \binom{k}{2}S_{k-2}(n) - \binom{k}{3}S_{k-3}(n) - \cdots - \binom{k}{k-1}S_1(n)\,. \quad (*)$$

Obtain simplified expressions for $S_3(n)$ and $S_4(n)$.

(ii) Explain, using $(*)$, why $S_k(n)$ is a polynomial of degree $k+1$ in n. Show that in this polynomial the constant term is zero and the sum of the coefficients is 1.